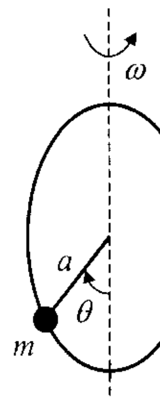
2001

Part I (May 16, 8:00AM-11:00AM) Classical Mechanics
Select Three out of Four Problems

1. A basketball player tosses a ball from 1.7 m above the court at an angle of 70° from the horizontal to sink the ball into the basket 3.3 m above the court and 4.0 m away. (a) With what speed must the ball be thrown to make the shot? (b) What is the maximum height above the court that the ball will reach? (c) At what speed will the ball enter the basket?

2. A ball is initially projected, without rotation, at a speed v_0 up a rough inclined plane at inclination θ. The coefficients of static and kinetic frictions are μ_s and μ_k, respectively. (a) Draw a free-body diagram and find the time t_1 at which pure rolling begins. (b) Draw a free-body diagram for time $t > t_1$, and find the time t_2 after which the ball comes momentarily to a stop. Assume that $\mu_s > 2\tan\theta/7$. Moment of inertia of a sphere about its CM is $2mR^2/5$.

3. A bead of mass m slides without friction on a circular hoop of radius a. The hoop lies in a vertical plane, and is constrained to rotation about a vertical diameter with angular frequency ω. (a) Write the Lagrangian that describes the motion of the bead. (b) Show that $\dot{\theta}\,\partial L/\partial\dot{\theta} - L = H$ is a constant of the motion. (c) Is H the total energy? Explain. (d) Using the constant of the motion H of part (b), qualitatively analyze the motion of the bead. Note that there is a difference in the motion depending on whether ω is greater than or less than critical value $\omega_c = \sqrt{g/a}$.



4. The Sun loses mass radially and isotropically (same in all directions) at a rate of about $\frac{dM}{dt} = -1.0x10^{-13} M$ per year, where M is the mass of the Sun, because of its radial flux of photons and particles (Solar Wind). (a) Which physical quantities related to the Earth's orbit are conserved during this process? Give physical reasoning. (b) Calculate the fractional effect $\frac{dr}{dt}\Big/ r$ on the radius of the Earth's orbit assuming that its orbit is circular. (c) State whether the orbit gets bigger or smaller with time.

2001

Part II (May 16,12:00noon-3:00PM) Classical Electromagnetic Theory
Select Three out of Four Problems

1. An infinite plane conducting surface occupying the $z = 0$ plane is grounded except for a square of side $2a$ centered at the origin. The square is held at a potential V_0. It is desired to find the potential everywhere in the $z \geq 0$ half space. (a) Write down (do not derive) the Green function for the problem. (b) Write down an integral expression for the potential in the $z \geq 0$ half space. Show the integrand and all the limits of integration explicitly. (c) Evaluate the leading term in the potential for points such that $r \gg a$.

2. (a) Consider a homogeneous, isotropic, nonmagnetic conducting medium characterized by dielectric permittivity $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega) \approx i\sigma/\omega$ where the imaginary part of the dielectric permittivity of the medium is large compared to its real part ($\varepsilon''(\omega) \gg \varepsilon'(\omega)$) and σ is the dc conductivity of the medium. Determine the form of monochromatic plane wave solutions of the Maxwell's equations when the propagation and attenuation occur in the same direction (homogeneous plane wave), say, parallel to the z direction. (b) Calculate the ratio of electric and magnetic energy densities in the medium and discuss the relative importance of electric and magnetic energies in highly conducting media.

3. A plane electromagnetic wave traveling in a medium with (real) dielectric constant ε_1 strikes the interface of another medium with (real) dielectric constant ε_2. Assume the media are nonmagnetic. (a) Express the indices of refraction of the media in terms of ε_1 and ε_2. (b) Using the appropriate continuity conditions at the boundary assuming that the wave polarization is perpendicular to the incident plane, derive Snell's law of refraction. (c) For *normal incidence*, use the boundary conditions for the fields to derive the values of the reflection and refraction coefficients.

4. (a) Consider a free electron moving perpendicular to a uniform magnetic field B_0 (Ignore the spin of the electron). Express the kinetic energy of the electron in terms of its orbital magnetic moment. (b) Now suppose that the strength of the magnetic field is slowly increased. What effect does this have on the orbital magnetic moment? (c) Now consider the motion of an electron moving in the following magnetic field which is symmetric with respect to the z axis: $B_\theta = 0$, $B_z = B(z)$, and $B_r = -(1/2)\, r\, dB(z)/dz$. Take the case where $B(z) = B_0\,(1+z^2/a^2)$. Determine the period of the oscillations along the z-axis of an electron moving in this magnetic field.

Part III (May 18, 8:00AM-11:30AM) Quantum Mechanics, Set #1
Select Three out of Four Problems

1. ψ_1 and ψ_2 are eigenfunctions of an electronic Hamiltonian $H^{(0)}$, with ε_1 and ε_2 as eigenvalues, respectively (i.e. $H^{(0)}\psi_1 = \varepsilon_1\psi_1$ and $H^{(0)}\psi_2 = \varepsilon_2\psi_2$, with $\varepsilon_1 > \varepsilon_2$). Let's denote a second electronic Hamiltonian H such as $H = H^{(0)} + \delta V$, where δV is a potential. We have $<\psi_1|\delta V|\psi_2> = <\psi_2|\delta V|\psi_1> = \varepsilon_s$ where ε_s is real. Let's also denote by ϕ_1 and ϕ_2 the eigenfunctions of H with E_1 and E_2 as eigenvalues, respectively. We will assume that ϕ_1 and ϕ_2 are linear combinations of ψ_1 and ψ_2, i.e. $\phi_1 = a\psi_1 + b\psi_2$ and $\phi_2 = c\psi_1 + d\psi_2$ (with a, b, c, and d real). (a) Express E_1 and E_2 in terms of ε_1, ε_2, and ε_s. (b) Express a, b, c, and d in terms of ε_1, E_1, E_2, and ε_s. (c) Express E_1 and E_2 in terms of ε_1, ε_2, and ε_s when $\varepsilon_s^2 \ll (\varepsilon_1-\varepsilon_2)^2$.

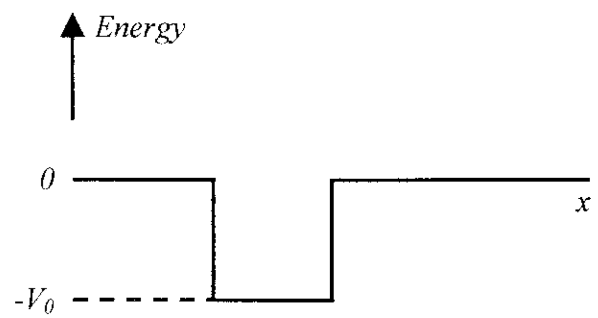
2. Using first-order perturbation theory, calculating the change in the energy levels of a harmonic oscillator which is perturbed by a potential gx^4. (Hint: express x in terms of the creation and annihilation operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right)$$

$$a^+ = \sqrt{\frac{m\omega}{2\hbar}}\left(x - i\frac{p}{m\omega}\right)$$

and recall that the harmonic oscillator eigenstates satisfy $a^+|n> = \sqrt{n+1}\,|n+1>$, $a\,|n> = \sqrt{n}\,|n-1>$.)

3. A quantum-mechanical particle approaches from the left a square potential dip of $-V_0$ like the one shown in the figure. Assume that the particle's energy $E > 0$. (a) What uniquely quantum-mechanical effect will take place that has no classical analog? (b) For what values of the energy is the transmission coefficient maximum? Give a physical interpretation of this result: what makes these energies "special"? (c) In real life the potential well would not have totally sharp edges. Discuss qualitatively what happens to the reflection coefficient as the edges become smoother. How should the particle's wavelength and the characteristic distance over which the potential changes be approximately related in order to maximize the effect discussed in (a)?



4. Consider two spin ½ particles (not identical) with no orbital motion. Their spin interaction is

$$\hat{H} = \hbar A \hat{\vec{S}}_1 \bullet \hat{\vec{S}}_2$$

At $t = 0$, particle 1 has spin up ($s_{1z} = +1/2$) and particle 2 has spin down ($s_{2z} = -1/2$). At $t > 0$, find (a) the probability of particle 1 spin up & particle 2 spin down; (b) the probability of particle 1 spin up and particle 2 spin up at the same time; (c) the probabilities of finding total spin equals to 1 ($S = 1$) and 0 ($S = 0$), respectively; (d) expectation values of $\hat{\vec{S}}_1$ and $\hat{\vec{S}}_2$ as a function of t.

1. A two-state quantum system evolves as follows in 5 sec.

State $|1\rangle$ becomes state $|1'\rangle = -\sqrt{3}/2\,|1\rangle - i/2\,|2\rangle$

State $|2\rangle$ becomes state $|2'\rangle = -i/2|1\rangle - \sqrt{3}/2|2\rangle$

(a) Derive a complete set of states that each would stay the same (except for a possible phase) at all times.

(b) Compute the energy level splitting assuming it is the lowest possible to achieve the 5 sec. evolution.

(c) Derive an expression for evolution of each state at a time of 1 sec.

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2. Early quantum theory and modern physics considered three fundamental distances or radii: the Bohr radius, the Compton or Dirac radius, and the classical radius of an electron.

(a) The Bohr radius r_{Bohr} is the most well known. Derive its formula in terms of *e, m, c*, h and ε_0 and approximate value of r_{Bohr} and the expected speed of the electron in units of *c*.

(b) The classical radius $r_{classical}$ of the electron involves a conjecture that rest energy is due to electrostatic $PE= e^2/(4\pi\varepsilon_0\, r_C)$. Derive this hypothetical $r_{classical}$.

(c) The Dirac-Compton radius r_{Dirac} is least known. Dirac theory shows something (The *"Zwitterbegung"*) inside an electron is always moving at *c*, the speed of light. Suppose this something moves around a circle at the transition frequency between the positive and negative rest energy levels $\pm mc^2$. Calculate the diameter $2r_{Dirac}$ of this hypothetical circle.

(d) Derive and compare ratios: $2r_{Dirac}/r_{Bohr}$ and $r_{classical}/2r_{Dirac}$. Is one or both of the ratios well known?

Use mks values $e = 1.6E\text{-}16C$, $m = 9.1E\text{-}31kg$, $c = 3E8m/s$, $h = 1.05E\text{-}34Js$ and $1/(4\pi\varepsilon_0)=9E9\ Jm^2/C^2$

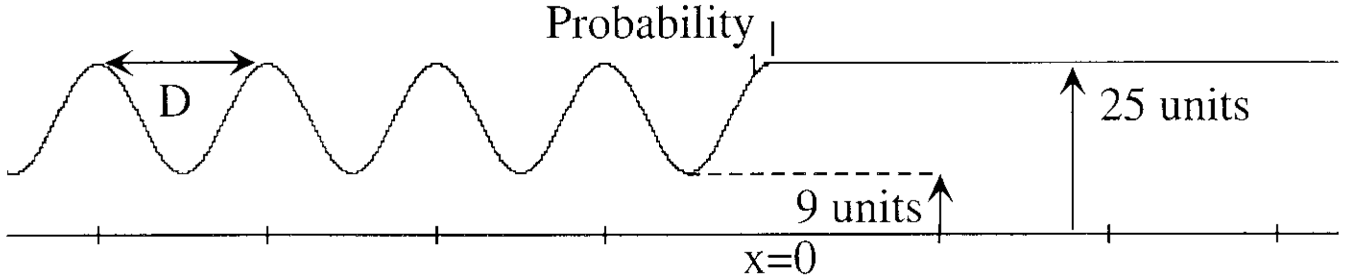
Fundamental quantum and electronic constants

e=1.602E-19 Coulomb.

m=9.11E-31 kilogram.

c= 3.00E+8 meters/second .

ℏ= 1.05E-34 Joule second = h/2π .



3. On the left side of an interface, a wave probability distribution has 25-unit-high peaks with 9-unit-high valleys separating each peak by *D nm*. (See Figure) A constant 25-unit-high distribution comes after $x=0$. This is observed when an electron beam of total energy $E = 25\ eV$ is propagated from the left $(x<0)$ toward an interface at $x=0$ where it partially reflects. The $(x>0)$-region to the right of $x=0$ has no electron sources, only sinks.

(a) How long is *D* in nanometers. (First show that a *1 eV* electron has a *1.23 nm* wavelength.)

(b) Does the $(x>0)$-region have higher or lower potential energy *V*? (Assume $(x<0)$-region is $V=0\ eV$.)

(c) What is the probability reflection coefficient *(Current reflected<-from x=0)/(Current into-> x=0)* ?

(d) How much higher (or lower) is the $(x>0)$-potential *V*? (Compute *V*.)

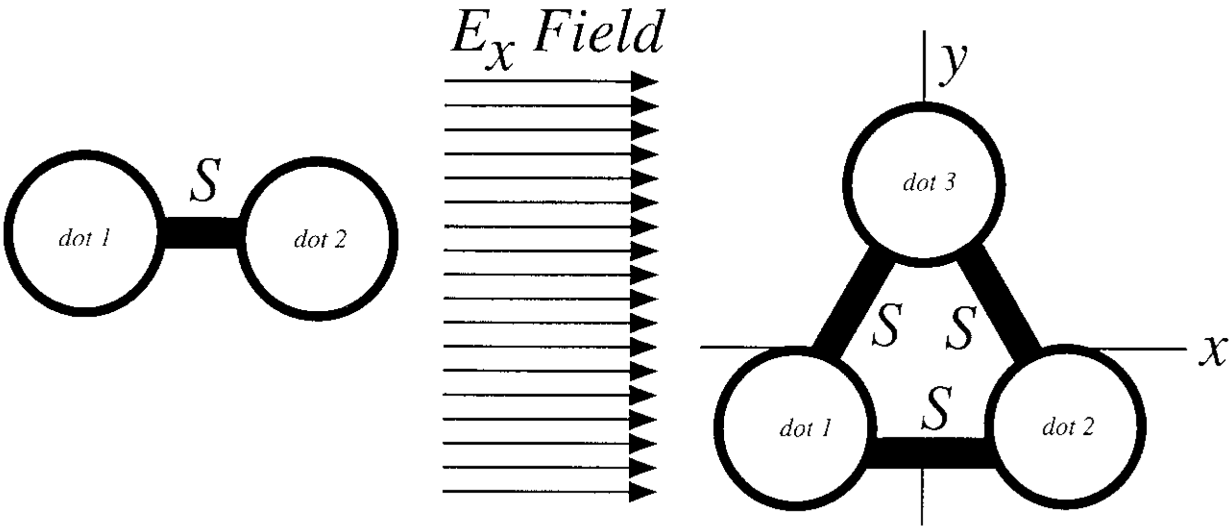


Fig. 2(a) Fig. 2(b)

4. Suppose a coupled C_2-symmetric 2-quantum-dot system (Fig. 2a.) is found to have a *1 MHz* resonance when excited by radiation polarized along its axis of coupling. Let identical dots and coupling connectors be used to make a C_3-symmetric 3-dot system (See Fig. 2b.)

(a) Write the time-dependent 2-by-2 matrix Schrodinger equation for the C_2 system using the minimum number of parameters to describe system and its E_x-field perturbation. Give solutions for the case of zero field and evaluate whichever parameters are determined by the observed *1 GHz* resonance. Compute time-evolution assuming 100% *(t=0)* probability localized on dot state #1 $|{}^{\#}1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ after time of *1/4μsec.* and *1/2μsec.*

(Give probability to be in dot state #2 at these times.)

(b) Write the time-dependent 3-by-3 matrix Schrodinger equation for the C_3 system using the minimum number of parameters employed in part (a) to describe system and its E_x-field perturbation for x-polarization. Give solutions for the case of zero field and compare the frequency(s) of its resonance(s) to that of the C_2 system. Compute time-evolution assuming 100% *(t=0)* probability localized on dot state #1 $|{}^{\#}1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ after time of *1/3μsec.*and *1/2μsec.*

(c) Compare power emitted to dipole radiation field at resonance for the two systems if each starts 100% in an excited zero-field eigenstate.
