2002 Ph.D. Candidacy Exam

Part I (May 15, 8:00 am—11:00 am) Classical Mechanics
Select three out of four problems

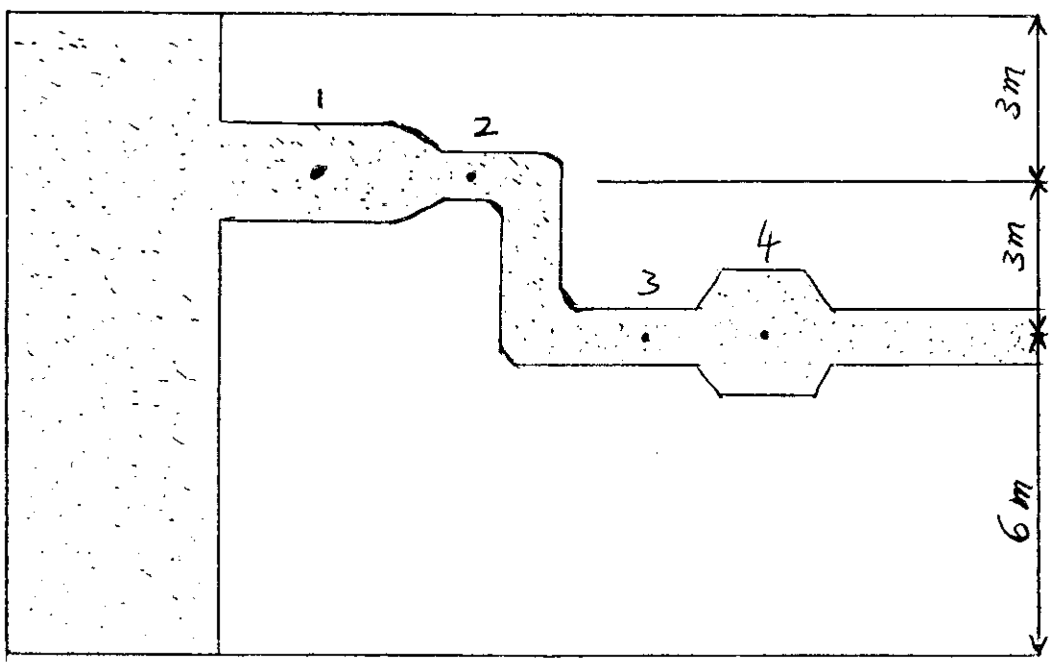
1. A solid sphere having a radius of 0.4 m and weighing 60 N, starts rolling (without slipping) from a height of 15.0 m down a long straight track titled at an angle of 30 degrees above the horizontal.
(a) Find the force of friction.
(b) Find the speed with which it leaves the track.
(c) Find the percent change in speed of (1) this sphere rolling to (2) this sphere moving down this incline after it is made to be smooth.

2. Let's consider a fluid that is flowing in a lake that varies in elevation as well as in cross-sectional area.
(a) Use the work-energy theorem to demonstrate the Bernouilli's equation, i.e.:

$$P+\frac{1}{2}\rho g y+\frac{1}{2}\rho v^2 = \text{constant}$$

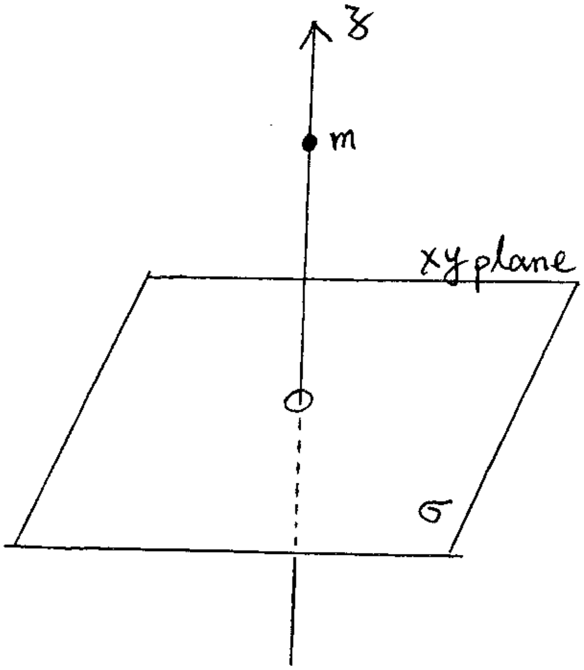
where P is the pressure, ρ is the density, y is the height, and v is the velocity.

(b) Water is discharged from the tank in the manner shown in the following figure. Use the Bernouilli equation and the continuity equation (i.e., Av=constant, where A is the cross-sectional area and v the velocity) to find at which point is the pressure the least, among the points 1, 2, 3, and 4.



3. A sphere of radius a is constrained to roll without slipping on the lower half of the inner surface of a hollow cylinder of inside radius R. Determine
(a) the Lagrangian,
(b) Lagrange's equations of motion,
(c) the force of constraint, and
(d) the frequency of small oscillations.

4. An isolated thin infinite plane mass-sheet (on the xy plane) with a mass-density of σ has a small hole in it, through which can pass a small point-mass. If the point-mass is placed a distance z above the hole and released from rest,
(a) describe the motion of the point-mass precisely,
(b) discuss the timescale of the motion, and
(c) state the maximum velocity of the point-mass.
Neglect all other gravitational fields (such as the earth's gravitational field).



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Part II (May 15, 12:00 pm—3:00 pm) Classical Electromagnetic Theory
Select three out of four problems

1. A coaxial capacitor is constructed from two concentric, hollow, very long metal cylinders. The inner cylinder has a radius a and a charge per meter of λ. The outer cylinder has an inner radius of b and is grounded. The space between the cylinder is filled with a dielectric with a relative permittivity $\kappa=\varepsilon/\varepsilon_0$ that is not constant. Instead, κ varies with radius such that the electric field everywhere in the dielectric is constant: $\vec{E} = E_0\hat{r}$. It is also known that $\kappa=3$ at r=b.

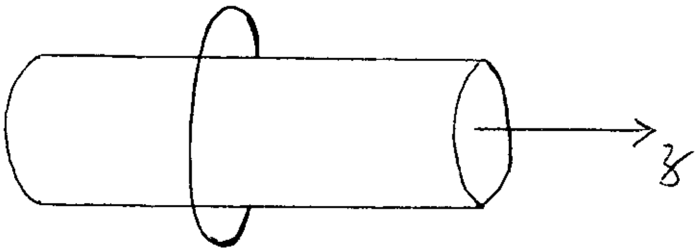
(a) Find E_0 as a function of λ, b, and ε_0.
(b) Find the expression for $\kappa=\kappa(r)$.
(c) Determine an expression for the capacitance per meter of this system: $C=C(a, b, \varepsilon_0)$.
(d) What is the dipole moment per volume

$$\vec{P} = P_r\hat{r} + P_\theta\hat{\theta} + P_Z\hat{z}\ .$$

(e) Obtain expressions for the free and bound surface charge densities at r=a in terms of λ, a, b.
(f) Obtain an expression for the bound volume charge density at $a \le r \le b$ in terms of λ, b, r.
(g) Compare the two expressions you obtained in part (e) in the limit that b>>a. Do your answers make sense? Explain.

2. A spherical surface of radius R is kept at the potential $V(\theta)=V_0\cos^2\theta$.
(a) Find the potential inside and outside the sphere.
(b) Find the field inside and outside the sphere.
(c) Is there charge on the surface r=R? Find how it is distributed. What is the net charge on the sphere?
(The first few Legendre polynomials are $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=(3x^2-1)/2$, $P_3(x)=(5x^3-3x)/2$, … …)

3. A cylinder of unknown material is placed concentrically within a ring of wire. The axes of both the cylinder and ring are along the z axis. The cylinder has radius 1.0 cm, and the wire ring has radius 1.5 cm. This system is subjected to a time-varying uniform magnetic field of the form $\vec{H}(t) = H_0 \cos\omega t\ \hat{k}$, where H_0=0.0001 A/m, and $\omega=2\pi\nu$, with ν=40 kHz. An EMF $\varepsilon(t) = \varepsilon_0 \sin\omega t$, with $\varepsilon_0=2.0\times10^{-8}$ volts is measured on the ring. What is the magnetic susceptibility χ of the unknown material?· Assume χ is scalar and uniform, and that demagnetization effects can be ignored.



4. A highly conducting medium with conductivity σ carries no net charge (ρ_e=0).

(a) Consider a monochromatic wave in this medium moving in the +z direction with the magnetic field given by

$$\vec{B}(z,t) = \hat{y}\, B_0\, e^{i(kz-\omega t)}.$$

Show that the amplitude of this wave decreases exponentially. Find the attenuation length (skin depth).

(b) For sea water, the conductivity is 5 $(\Omega\text{-m})^{-1}$. Calculate the attenuation length at radio wave frequencies ($\omega = 2\pi \times 5 \times 10^5\ \text{sec}^{-1}$). What can you conclude about communicating with sub-merged submarines using radio waves. For sea water, you can take $\mu=\mu_0$ and the real part of ε to be ε_0.

2002 Ph.D. Candidacy Exam

Part III (May 17, 8:00 am—11:30 am) Quantum Mechanics, set #1
Select three out of four problems

1. Solve the Schrodinger equation for a particle constrained to move on a circle of radius R. Derive eigenvalues and normalized eigenfunctions of E and L_z. Explain physical reasons for any degeneracy you find.
   Note, the Laplacian in cylindrical coordinates is given by

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}$$

2. The standard matrix representation of the angular momentum operators for angular momentum one-half are:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Is the matrix S_x+S_z hermetian?
(b) Find the eigenvalues of the matrix S_x+S_z.
(c) Find the eigenvectors of the matriz S_x+S_z.

3. Consider a free particle in one dimension
(a) Write down the Hamiltonian in terms of the operators x and p.
(b) Using the result

$$i\hbar\frac{d}{dt} < A >=< [A,H] >$$

valid for an arbitrary operator A (provided it does not depend explicitly on t), derive equations of motion for $<p>$ and $<p^2>$, and show that the variance of p, defined as $\Delta^2 p=<p^2>-<p>^2$, is a constant (independent of time).
(c) In the same way, derive equations of motion for $<x>$ and $<x^2>$ (plus any other auxiliary variables you may need) and integrate them to show that the variance of x, defined as $\Delta^2 x=<x^2>-<x>^2$, is given by the following function of time:

$$\Delta^2 x(t) = \Delta^2 x(0) + \frac{1}{m}\left(< xp + px >_0 -2 < x >_0 < p >_0\right)t + \frac{\Delta^2 p}{m^2}t^2 \quad , \quad (1)$$

where the subscript "0" denotes expectation values taken at the initial time t=0.
(d) Given a physical interpretation of Eq. (1) (for simplicity, consider only the case where the position and momentum are initially uncorrelated, so that the second term on the right-hand side of Eq. (1) vanishes).

4. Consider a one-dimensional harmonic oscillator with unit mass ($m=1$) and natural frequency ω_0. The Hamiltonian for such an oscillator can be written in each of the following forms:

$$\hat{H}=\frac{1}{2}\left(\hat{p}^2+\omega_0^2\hat{q}^2\right)=\hbar\omega_0\left(\hat{a}^+\hat{a}+\frac{1}{2}\right)$$

where

$$\left[\hat{q},\hat{p}\right]=i\hbar \text{ and } \hat{a}=\sqrt{\frac{\omega_0}{2\hbar}}\left(\hat{q}+i\frac{\hat{p}}{\omega_0}\right)$$

The energy eigenstates for the oscillator form the usual number state basis

$$\hat{H}|n\rangle=\hbar\omega_0\left(n+\frac{1}{2}\right)|n\rangle$$

The ground state configuration space wave function for this system is

$$\Psi_0=\langle q|0\rangle=\left(\frac{\omega_0}{\pi\hbar}\right)^{1/4}e^{-\frac{\omega_0 q^2}{2\hbar}}$$

(a) By considering the commutator $\left[\hat{a},\hat{H}\right]$, show that the action of the operator $\hat{a}$ is to remove one quantum of energy from the oscillator.

(b) Consider the superposition state $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$ where $|\alpha|^2+|\beta|^2=1$.

(i) Write down the explicit form of the normalized configuration space wave function $\langle q|\psi\rangle$.

(ii) Calculate the uncertainties in position and momentum as functions of the parameters α and β for this state. Are there and values of α and β for which this is a minimum uncertainty state? If so, which values?

(c) Consider the state $|\delta q\rangle=e^{-i\frac{\hat{p}\delta q}{\hbar}}|0\rangle$ where δq is a real parameter with the dimension of length.

(i) Express this state in the number state basis. You may use without proof the following identity:
If $\hat{A}$ and $\hat{B}$ are operators such that $\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]=\left[\hat{B},\left[\hat{A},\hat{B}\right]\right]=0$
Then

$$e^{\hat{A}}e^{\hat{B}}=e^{\hat{A}+\hat{B}+\frac{1}{2}\left[\hat{A},\hat{B}\right]}$$

(ii) Show that $|\delta q\rangle$ is a right eigenstate of the operator $\hat{a}$, that is, show that $\hat{a}|\delta q\rangle=z|\delta q\rangle$, where z is the eigenvalue of $\hat{a}$.

(iii) $|\delta q\rangle$ is often called a "displaced vacuum state." Why is this a fitting name?

2002 Ph.D Candidacy Exam (May 17- 8:00 AM - 11:30 AM)

Part III Quantum Mechanics - ~~Section~~ set #2 (Select 3 out of 4 problems)

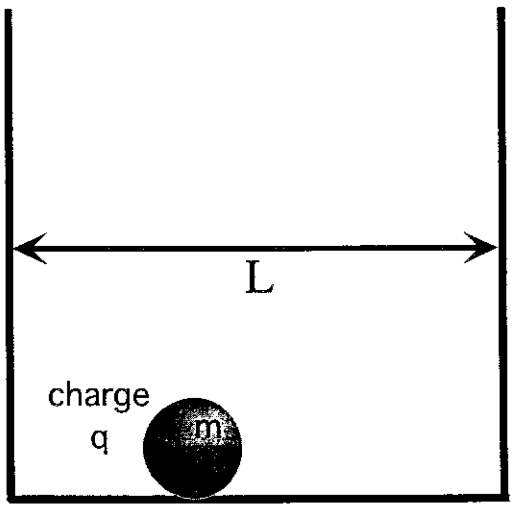
1. A two-state quantum system evolves as follows in 5 sec.

State $|1\rangle$ becomes state $|1'\rangle = -\sqrt{3}/2\,|1\rangle - i/2\,|2\rangle$

State $|2\rangle$ becomes state $|2'\rangle = -i/2|1\rangle - \sqrt{3}/2|2\rangle$

(a) Derive a complete set of states that each would stay the same (except for a possible phase) at all times.
(b) Compute the energy level splitting assuming it is the lowest possible to achieve the 5 sec. evolution.
(c) Derive an expression for evolution of each state at a time of 1 sec.

2.



~~Quantum Mechanics 0 (alternate)~~

Consider a particle of mass m and charge q in a one-dimensional box of length L, that is, let it be in an infinite square-well potential as indicated in the Figure.

(a) Calculate the energy of the two lowest stationary states according to standard quantum mechanics. Give energy values and normalized wavefunctions.

(b) Suppose the particle has probability P_1 to be in the first or lowest state and probability P_2 to be in the second or next lowest level where $1 = P_1 + P_2$. Calculate the expected energy of the particle for a 50-50 combination ($P_1 = P_2$).

(c) Calculate the expected position $\langle x(t)\rangle$ and electric dipole moment $\langle p(t)\rangle$ as a function of constants m, q, L, $\hbar$, and probabilities P_1 and P_2.

(d) Use your result from (c) to tell what P_1 and P_2 give the maximum $\langle x\rangle$ and what that value is as a fraction of length L. (Numerical answer only will be accepted for this part.)

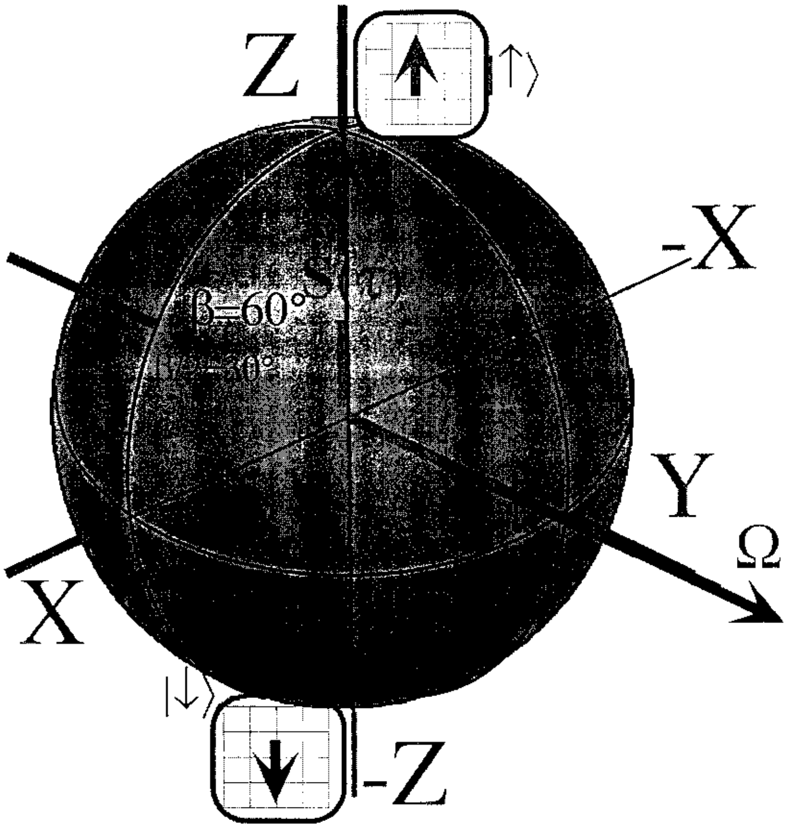


Fig. 1

3. Suppose an NMR spin system is initially in a state

$$|\Psi(0)\rangle = \frac{\sqrt{3}}{2}|\uparrow\rangle + \frac{1}{2}|\downarrow\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \quad (3.1)$$

(a) Find a constant Hamiltonian **H** which will drive this state thru spin-up $|\uparrow\rangle$ in a given time τ.

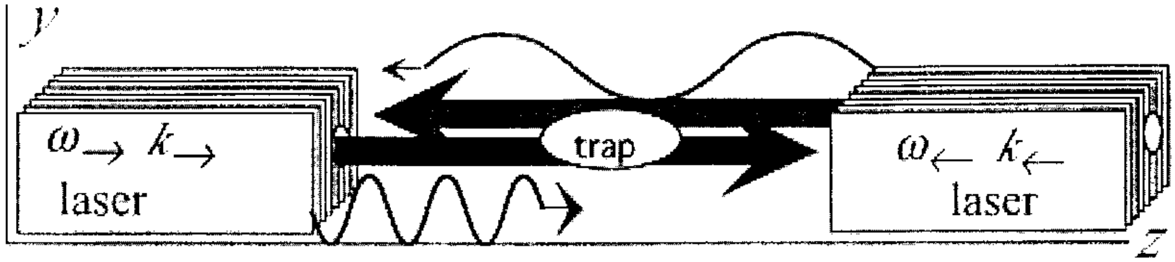
$$\left|\langle\uparrow|\Psi(\tau)\rangle\right|^2 = 1 \quad \text{but: } \left|\langle\uparrow|\Psi(t)\rangle\right|^2 \neq 1 \text{ for : } t < \tau \quad (3.2)$$

Many correct answers exist but full credit is one which requires the <u>least</u> energy.

(b) Give the eigenkets and energy eigenvalues of the Hamiltonian resulting from (a) in terms of τ and $\hbar$ and sketch an energy level diagram.

(c) Give a formula for the angular frequency of radiation in terms of τ and $\hbar$ that might be observed as this state and Hamiltonian are allowed to time-evolve.

(d) What is the <u>maximum</u> energy or frequency of radiation that can result from (3.1-2) above.



4. Suppose two lasers fixed to an optical table emit counter-propagating light (photons) along the beam z-axis into an optical atom trap. The right pointing laser emits y-polarized light of amplitude $E_{\rightarrow}$(Volt/m), frequency $\omega_{\rightarrow} = 2\pi\nu_{\rightarrow}$(per sec.) and wavevector $k_{\rightarrow}$ (per m.) while the left-pointing laser emits y-polarized light of amplitude $E_{\leftarrow}$, frequency $\omega_{\leftarrow} = 2\pi\nu_{\leftarrow}$, and wavevector $k_{\leftarrow}$. A wavefunction $\Psi(z,t)$ exists at each point z and time t so we may compute the relative probability $P(z,t)\, dz\, dt$ for an atom to capture a photon from either direction. The right pointing laser emits $N_{\rightarrow}$ photons per second and the left pointing laser emits $N_{\leftarrow}$ photons per second (average).

The main results are parts (e) and (f). The other parts may help in solving these.

Assume the atom is stationary relative to the laser:
(a) Write the wavefunction $\Psi(z,t)$ and probability $P(z,t)$. Note max-min values of $P(z,t)$.
(b) Give $E_{\rightarrow}$ and $k_{\rightarrow}$ in terms of $N_{\rightarrow}$ and $\omega_{\rightarrow}$ using fundamental constants $\hbar$, c, and ε_0.
(c) Give $E_{\leftarrow}$ and $k_{\leftarrow}$ in terms of $N_{\leftarrow}$ and $\omega_{\leftarrow}$ using fundamental constants $\hbar$, c, and ε_0.
(d) Derive a velocity formula for the nodes or anti-nodes of probability $P(z,t)$ in terms of frequencies $\omega_{\rightarrow}$ and $\omega_{\leftarrow}$ and (if necessary) time t and amplitudes $E_{\rightarrow}$ and $E_{\leftarrow}$. Compute numerical value for $\omega_{\rightarrow} = 4c$ and $\omega_{\leftarrow} = 1c$ and $E_{\rightarrow} = 0.5 = E_{\leftarrow}$.
(e) Derive a formula or formulas for the velocity of electric field zero-points ($\mathrm{Re}\Psi(z,t)=0$) in terms of frequencies $\omega_{\rightarrow}$ and $\omega_{\leftarrow}$ for equal but opposite amplitudes $E_{\rightarrow} = E_0 = -E_{\leftarrow}$. Compute numerical values for $\omega_{\rightarrow} = 4c$ and $\omega_{\leftarrow} = 1c$ asuming $E_{\rightarrow} = 0.5 = -E_{\leftarrow}$.
(f) Derive a formula for the velocity of the real electric field zero-points ($\mathrm{Re}\Psi(z,t)=0$) in terms of amplitudes $E_{\rightarrow}$ and $E_{\leftarrow}$ and time t for equal frequencies $\omega_{\rightarrow} = 2c = \omega_{\leftarrow}$. Compute the numerical values for maximum and minmum speeds for $E_{\rightarrow} = 4$ and $E_{\leftarrow} = 1$. (You may express in terms of the *standing wave ratio* as $SWR = (E_{\rightarrow} - E_{\leftarrow})/(E_{\rightarrow} + E_{\leftarrow})$.)
