Department of Physics

CANDIDACY EXAMINATION (2003)
Part I

(Classical Mechanics)

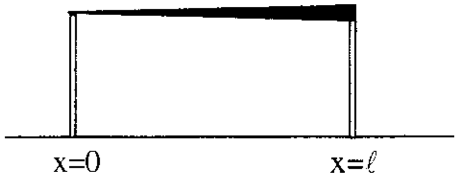
TIME: 8:00am — 11:00am, Wednesday, May 14, 2003

1. Answer a total of three out of four questions in Part I.

2. Use only one side of the answer paper. Write your code on each sheet in the top, right-hand corner and your problem number in the top left-hand corner.

3. Hand in each Problem Solution as a unit.

1. A non-uniform plank of mass m, and length ℓ rests on two thin supports, as shown on the right. The linear mass density of the plank varies as $\mu(x)=\mu_0 x$.



(a) Find the position of the center-of-mass of the plank. (10 points)

(b) Find the moment of inertia of the plank about a vertical axis through x=0. (10 points)

(c) Find the forces F1 and F2 applied by the supports on the plank. (20 points)

(d) The support at $x=\ell$ is now suddenly removed. Find the new force F'_1 applied by the support at x=0, and the acceleration of the center-of-mass of the plank, immediately after the support at $x=\ell$ is removed. (60 points)

2. An almost spherical planet of uniform density, mass M_s, and radius R_s is spinning with a rotational velocity ω_s. This planet has a moon (of negligible mass M_0) orbiting it in a circular orbit of radius R_0, in the same direction as the planet's spin. Due to the tidal gravitational interaction, the spin of the planet is slowing down at a rate $\dot{\omega}_s$. Assume that the radius of the planet remains constant, but the radius of the moon's orbit varies with time.

(a) What general physical principle may be used to analyze the physics of this situation?
(20 points)

(b) Calculate the rate of change of the moon's orbital radius, $\dot{R}_0$.
(60 points)

(c) Is the moon's orbit getting bigger or smaller?
(20 points)

3. A particle with mass M is moving along the x–axis in the potential $V(x) = -kxe^{-ax}$, where k and a are positive constants.

(a) Find the equilibrium position and the period of small oscillation around the equilibrium position. (60 points)

(b) What will be the motion of the particle if k and a are both negative? (20 points)

(c) What will happen if one of the constants (k and a) is positive and another is negative? (20 points)

4. Let's mimic a crystal by a one-dimensional and infinite chain of identical atoms of mass M. The distance between two nearest neighbors is denoted as 'a,' and we assume that nearest atoms are connected by massless springs having a spring constant to be denoted as 'K.' Each atom is indexed by an integer n, where n ranges between $-\infty$ to $+\infty$ (since the chain is infinite).

Let's denote '$u(n)$' the displacement of the atom indexed by n with respect to its 0-Kelvin equilibrium position.

(a) Use the equation of motions to find the relationship between $\frac{d^2u(n)}{dt^2}$, (where t is the time), $u(n)$, $u(n+1)$, and $u(n-1)$.
(40 points)

(b) We are seeking solutions of the form:
(40 points)

$$u(n) = \mathrm{A}\exp(i(kna-\omega t)) \qquad \text{(Eq.1)}$$

where A is a constant, k is a wave-vector, and ω is an angular frequency.

Use Question 1 and Equation 1 to prove that $\omega = 2\sqrt{\frac{K}{M}}\,|\sin(\frac{ka}{2})|$

(c) Express the sound velocity in this crystal in terms of K, M and a.
(20 points)

Department of Physics

Part II

(Classical Electromagnetic Theory, including relativistic formulation)

TIME: Noon—3:00pm, Wednesday, May 14, 2003

1. Answer a total of three out of four questions in Part II

2. Use only one side of the answer paper. Write your code on each sheet in the top, right-hand corner and your problem number in the top left-hand corner.

3. Hand in each Problem Solution as a unit.

1. An alternating current $I=I_0\cos(\omega t)$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a.

(a) In what direction does the induced electric field point (radial, circumferential, or longitudinal)? (20 points)

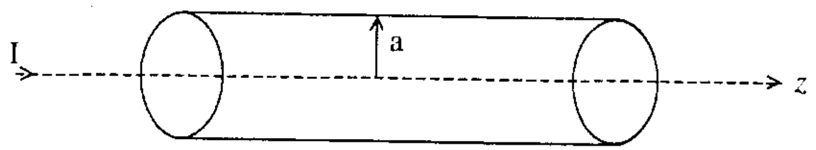
(b) Assuming that the field goes to zero as ρ goes to infinity, find $\vec{E}(\rho,t)$. (ρ is the radial component of the position vector when using the spherical coordinates.) (20 points)

(c) Find the displacement current density $\vec{J}_d$. (20 points)

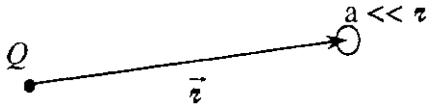
(d) Integrate it to get the total displacement current. (20 points)

$$I_d = \iint \vec{J}_d \cdot d\vec{a}.$$

(e) Compare I_d and I. (What is their ratio?) If the outer cylinder is, say, 2mm in diameter, how high would the frequency have to be, for I_d to be 1% of I? (20 points)



2. A small dielectric particle of size a, and dielectric permittivity ε is located at a distance r >> a from a point charge Q located at the origin. Find the force on the particle as a function of distance r. (100 points)



3. The electric vector of a plane electromagnetic wave is given by

$$\vec{E}(z,t) = \hat{x}\frac{E_0}{\sqrt{2}}e^{i(kz-\omega t)} + \hat{y}\frac{E_0}{\sqrt{2}}e^{i(kz-\omega t+\pi/2)}$$

(a) What is the state of polarization of this wave? What is the magnetic field of this wave? (60 points)

(b) Assuming that this is the field of a 5.0mW laser beam of diameter 1.0 mm, find the value of the field amplitude E_0. (40 points)

4. A ring of charge of radius R, mass m, and total charge q, is mounted (using insulated, uncharged spokes of negligible mass) on an axle around which it is free to turn without friction. Initially, the ring is at rest; then a uniform magnetic field, parallel to the axis, is established. The initial value of the magnetic induction is zero, and the final value is B. What is the final value of the ring's angular velocity? (Hint: use Faraday's law. You do *not* need to know the precise rate at which B grows, only its initial and final value.)
(100 points)

Department of Physics

Part III – Section *A* | ***or*** | Part III—Section *B*

Quantum Mechanics

TIME: 8:00am-11:30am, Friday, May 16, 2003

1. You must choose either:

   Part III—Section *A*

   or

   Part III—Section *B*

   by 8:30am.

2. Return the Section (i.e., *A* or *B*) that you choose NOT to do.

3. Answer a total of three out of four questions in the section of Part III you have chosen (i.e., *A* or *B*).

4. Use only one side of the answer paper. Write your code on each sheet in the top, right-hand corner and your problem number in the top left-hand corner.

5. Hand in each Problem Solution as a unit.

Part III—Section A

1. A harmonic oscillator wavefunction at t=0 is given by

$$\Psi(x,t=0)=\frac{\sqrt{10}}{5}\Psi_1(x)+\frac{3i}{5}\Psi_3(x)-\frac{\sqrt{6}}{5}e^{i\alpha}\Psi_4(x)$$

Where $\Psi_n(x)$ is an eigenfunction of energy corresponding to eigenvalues $E_n=(n+\frac{1}{2})\hbar\omega$.

(a) If a measurement of energy is carried out, what value is most likely to be measured?

(b) What is the probability that $E=9\hbar\omega/2$ will be measured?

(c) If $E=\frac{9}{2}\hbar\omega$ is measured, what will be the oscillator's wavefunction subsequent to this measurement?

(d) If there are, let us say, 100 identical systems, all described by the same wavefunction $\Psi(x, t=0)$ given above, and a measurement of energy is carried out on all the systems, what will be the <E> average value of the energy measured?

(e) If there is one system described by the wavefunction $\Psi(x, t=0)$ given above, and 100 successive measurements of energy are carried out on the same system, what can you say about average value of the measured energy?

(f) If no measurement on $\Psi(x, t=0)$ is carried out, write the wavefunction at $t>0$.

(g) For $t>0$, does <E> depend on t?

(h) For $t>0$, does <x> depend on t?

Points: All questions carry equal weight.

2. A particle is moving freely inside a one-dimensional infinite potential box with walls at x=0 and x=a, that is

$$V(x) = \begin{cases} \infty (x \le 0) \\ (0 < x < a) \\ \infty (x \ge a) \end{cases}$$

(a) What are the energy eigenfunctions (normalized) and eigenvalues for the particle? (50 points)

(b) If the electron is initially in the ground state (n=1) of the box, and if we suddenly quadruple the size of the box (i.e., the right-hand side wall is moved instantaneously from x=a to x=4a), calculate the probability of finding the electron in:

1) the ground state of the new box.
2) the first excited state of the new box.

(50 points)

3. Consider the Hamiltonian for a 3-level system:

$$H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

when expressed in an orthonormal basis. The system is acted upon by a perturbation:

$$H' = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$ where λ is a small parameter.

(a) Find the energy levels of the perturbed system to second order in λ, using perturbation theory. (50 points)

(b) Find the exact energy levels and compare with the approximate values of Part (a). (50 points)

4. A particle is described by the wavefunction

$$\psi(x, y, z) = \mathrm{N}(xy + yz + zx)\exp(-\alpha(x^2 + y^2 + z^2))$$

where N is a suitable normalization constant.

(a) Express ψ as a product of a radial wavefunction (a function of the variable r only) and an appropriate superposition of spherical harmonics (functions of θ and ϕ). (50 points)

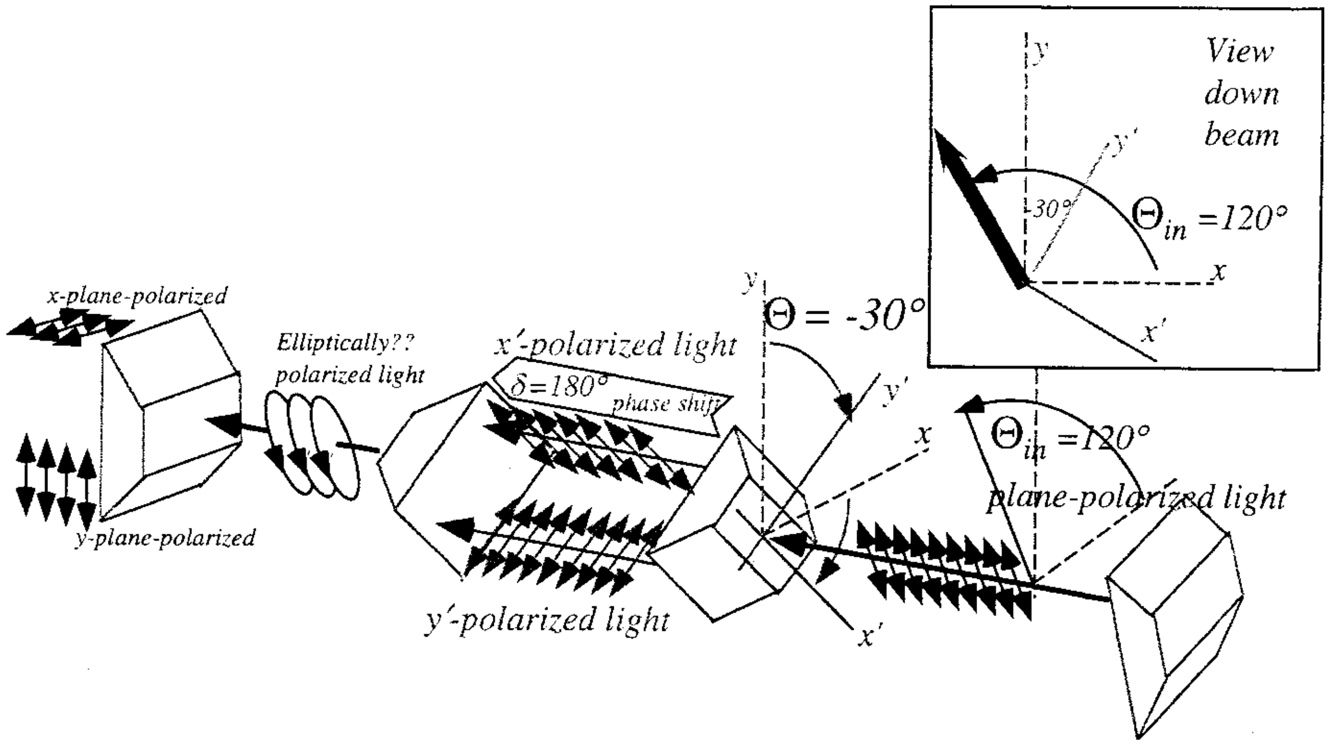
NOTE: The first few spherical harmonics are:

$$Y_{00} = (1/4\pi)^{1/2},\ Y_{10} = (3/4\pi)^{1/2}\cos\theta,\ Y_{1\pm1} = \mp(3/8\pi)^{1/2}\sin\theta e^{\pm i\varphi},$$
$$Y_{20} = (5/16\pi)^{1/2}(3\cos^2\theta - 1),\ Y_{2,\pm1} = \mp(15/8\pi)^{1/2}\cos\theta\sin\theta e^{\pm i\varphi},$$
$$Y_{2,\pm2} = (15/32\pi)^{1/2}\sin^2\theta e^{\pm 2i\varphi}.$$

(b) What is the probability that a measurement of $\mathbf{L}^2$ and L_z yields the results $6\hbar^2$ and $\hbar$, respectively? (Note: You don't need to complete part (a) in order to be able to answer this, but it should help.) (50 points)

Part III Section B

1.



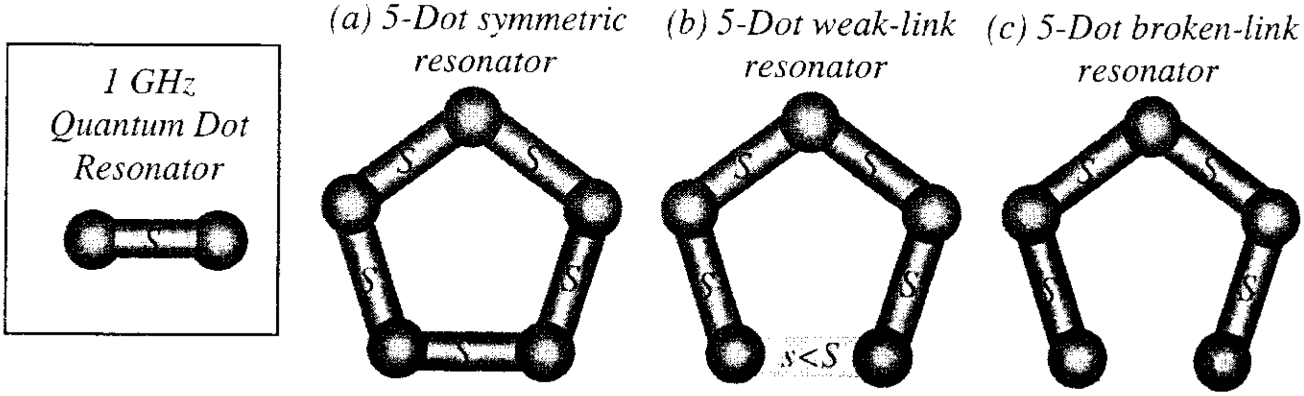
Consisder initial $+120°$-plane-polarization beam of 160 photons coming in from the right toward an active analyzer, that is, one that advances the phase of its high $-30°$-x'-plane beam ahead by phase angle ($180°$) relative to its low $-+60°$-y'-plane beam as indicated above the analyzer in the figure. As a result the initial $120°$-plane-polarization may be transformed into some elliptical polarization output beam.

Finally the output beam is analyzed into x or $0°$-plane or y or $90°$-plane photons at the extreme left hand side of the experiment sketched above.

(50 pts)
(a) Describe the output state and derive the expected number of photons counted in each output channel x or y. Is analyzer output elliptical? How elliptical?
(25 pts)
(b) Assuming instead that the high $-30°$- x'-plane beam is *blocked* inside the center analyzer, derive the number of photon counts in each output channel x or y.
(25 pts)
(c) Assuming instead that the high $+60°$-y'-plane beam is *counted*† inside the center analyzer derive the number of photons counted in each output channel x or y.

† *Counting* means observing if photon passed by but otherwise not impeding its passage.

2.



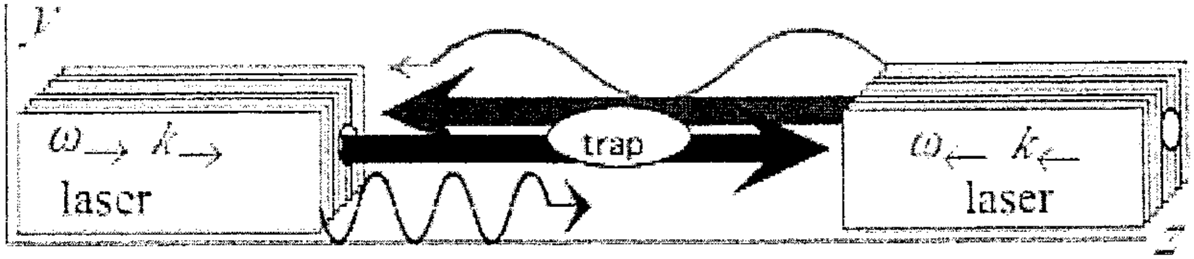
Five identical quantum dots are originally connected by five identical evanescent tunneling "weak-links" as shown in Figure (a) above. It is known that a pair of quantum dots linked by a *single* path of tunneling amplitude S (See inset.) has a *1.0 GHz* resonance as its lowest spectral frequency.

Subsequently, the bottom link weakens. First, as sketched in Figure (b), its tunneling amplitude $s<S$ varies only slightly. Later, as sketched in Figure (c) it goes to zero, that is, it is removed entirely.

(40 pts)
(a) Derive and solve a model Hamiltonian matrix for the Figure (a) device. This should include a matrix representation and its eigenvectors and eigenvalues. These may be given in terms of the tunneling amplitude S, but numerical value of S in Hz should be given, as well. A sketch of the energy level diagram would help. Also, the radiation spectral frequencies one might observe should be indicated or discused.

(20 pts)
(b) Discuss how the Hamiltonian and energy levels are changed by a reduced lower link of Figure (b).

(40 pts)
(c) Discuss how the Hamiltonian and energy levels are changed by a removing the lower link as shown in Figure (c). Give eigenvectors and eigenvalues. Discuss resulting dipole allowed spectra.



3. Suppose two lasers fixed to an optical table emit counter-propagating light (photons) along the beam z-axis into an optical atom trap. The right pointing laser emits y-polarized light of amplitude $E_{\rightarrow}$(Volt/m), frequency $\omega_{\rightarrow} = 2\pi\nu_{\rightarrow}$(per sec.) and wavevector $k_{\rightarrow}$ (per m.) while the left-pointing laser emits y-polarized light of amplitude $E_{\leftarrow}$, frequency $\omega_{\leftarrow} = 2\pi\nu_{\leftarrow}$, and wavevector $k_{\leftarrow}$. A wavefunction $\Psi(z,t)$ exists at each point z and time t so we may compute the relative probability $P(z,t)\, dz\, dt$ for an atom to capture a photon from either direction. The right pointing laser emits $N_{\rightarrow}$ photons per second and the left pointing laser emits $N_{\leftarrow}$ photons per second (average). The main results are parts (e-f). The other parts may help in solving these. Assume the atom is stationary relative to the laser:

(a) *10 pts* Write the wavefunction $\Psi(z,t)$ and probability $P(z,t)$. Note max-min values of $P(z,t)$.

(b) *10 pts* Give $E_{\rightarrow}$ and $k_{\rightarrow}$ in terms of $N_{\rightarrow}$ and $\omega_{\rightarrow}$ using fundamental constants $\hbar$, c, and ε_0.

(c) *10 pts* Give $E_{\leftarrow}$ and $k_{\leftarrow}$ in terms of $N_{\leftarrow}$ and $\omega_{\leftarrow}$ using fundamental constants $\hbar$, c, and ε_0.

(d) *10 pts* Derive a velocity formula for the nodes or anti-nodes of probability $P(z,t)$ in terms of frequencies $\omega_{\rightarrow}$ and $\omega_{\leftarrow}$ and (if necessary) time t and amplitudes $E_{\rightarrow}$ and $E_{\leftarrow}$. Compute numerical value for $\omega_{\rightarrow} = 4c$ and $\omega_{\leftarrow} = 1c$ and $E_{\rightarrow} = 0.5 = E_{\leftarrow}$.

(e) *10 pts* Derive a formula or formulas for the velocity of electric field zero-points ($\text{Re}\Psi(z,t)=0$) in terms of frequencies $\omega_{\rightarrow}$ and $\omega_{\leftarrow}$ for equal but opposite amplitudes $E_{\rightarrow} = E_0 = -E_{\leftarrow}$. Compute numerical values for $\omega_{\rightarrow} = 4c$ and $\omega_{\leftarrow} = 1c$ asuming $E_{\rightarrow} = 0.5 = -E_{\leftarrow}$.

(f) *10 pts* Derive a formula for the velocity of the real electric field zero-points ($\text{Re}\Psi(z,t)=0$) in terms of amplitudes $E_{\rightarrow}$ and $E_{\leftarrow}$ and time t for equal frequencies $\omega_{\rightarrow} = 2c = \omega_{\leftarrow}$. Compute the numerical values for maximum and minmum speeds for $E_{\rightarrow} = 4$ and $E_{\leftarrow} = 1$. Express in terms of the *standing wave ratio* as $SWR = (E_{\rightarrow} - E_{\leftarrow})/(E_{\rightarrow} + E_{\leftarrow})$.

Assume an atom in a (z',t')-frame moving right to left relative to laser (z,t)-frame along beam z at speed u sees amplitudes $E'_{\rightarrow}$, $E'_{\leftarrow}$, frequencies $\omega'_{\rightarrow}$, $\omega'_{\leftarrow}$ and wavevectors $k'_{\rightarrow}$, $k'_{\leftarrow}$. Assume lasers are identical so $\omega_{\rightarrow} = 2c = \omega_{\leftarrow}$ and $E_{\rightarrow} = 0.5 = -E_{\leftarrow}$.

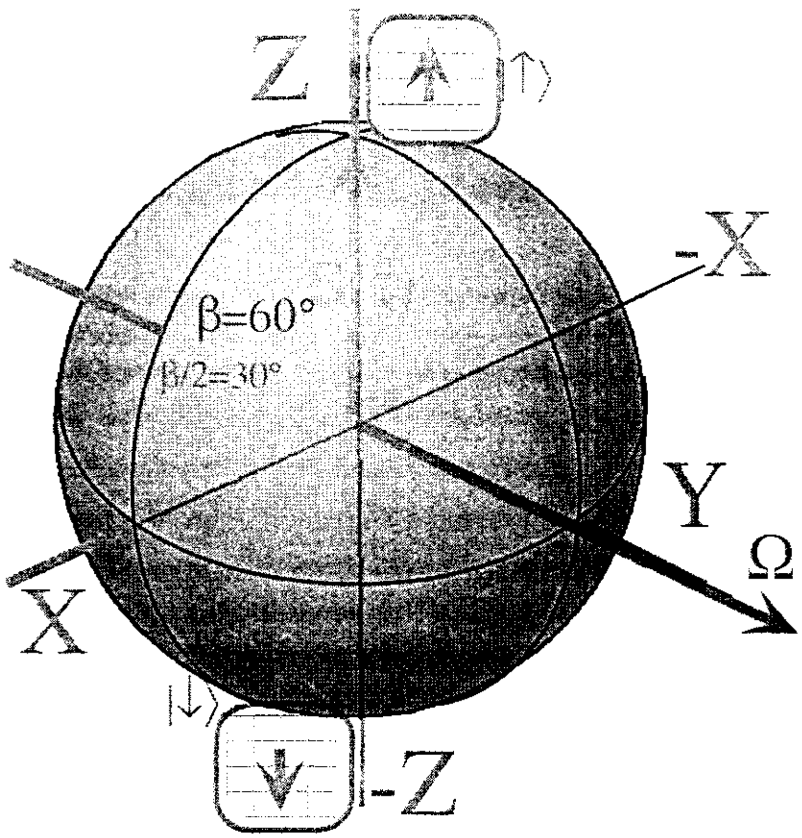
(g) *10 pt* Derive $\omega'_{\rightarrow}$ and $k'_{\rightarrow}$ in terms of $\beta = u/c$ and $\omega_{\rightarrow} = 2c = \omega_{\leftarrow}$ and numerically for $\beta = 3/5$.

(h) *10 pt* Derive $\omega'_{\leftarrow}$ and $k'_{\leftarrow}$ in terms of $\beta = u/c$ and $\omega_{\rightarrow} = 2c = \omega_{\leftarrow}$ and numerically for $\beta = 3/5$.

(i) *10 pt* Derive $N'_{\rightarrow}$ and $N'_{\leftarrow}$ in terms of $\beta = u/c$ and $N_{\rightarrow}$ and $N_{\leftarrow}$.

(j) *10 pt* Derive $E'_{\rightarrow}$ and $E'_{\leftarrow}$ in terms of $\beta = u/c$ and $E_{\rightarrow}$ and $E_{\leftarrow}$.

(k) *10 pt* Define the *standing wave ratio* as $SWR = (E_{\rightarrow} - E_{\leftarrow})/(E_{\rightarrow} + E_{\leftarrow})$ and derive SWR' seen by atom in terms of the SWR made by lasers and relativity parameter $\beta = u/c$.



4. Suppose an NMR spin system is initially in a state

$$|\Psi(0)\rangle = \frac{\sqrt{3}}{2}|\uparrow\rangle + \frac{1}{2}|\downarrow\rangle = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \qquad \text{(Requirement A)}$$

(a) *25 pts* Find a constant Hamiltonian **H** which will drive this state thru spin-up $|\uparrow\rangle$ in a given time τ.

$$|\langle\uparrow|\Psi(\tau)\rangle|^2 = 1 \quad \text{but: } |\langle\uparrow|\Psi(t)\rangle|^2 \neq 1 \text{ for : } t < \tau \qquad \text{(Requirement B)}$$

Many correct answers exist but full credit is one which requires the <u>least</u> energy.

(b) *35 pts* Give the eigenkets and energy eigenvalues of the Hamiltonian resulting from (a) in terms of τ and $\hbar$ and sketch an energy level diagram.

(c) *20 pts* Give a formula for the angular frequency of radiation in terms of τ and $\hbar$ that might be observed as this state and Hamiltonian are allowed to time-evolve.

(d) *20 pts* What is the <u>maximum</u> energy or frequency of radiation that can result from the requirements (A-B) given above.