2007 Ph.D. Candidacy examination
Quantum Mechanics
Jan. 11, 2007, 9:00AM-1:00PM

General instructions:

(i) Please do four problems of your choice. No more, no less.

(ii) Explain briefly your approach before solving each problem (to which 5 points will be given).

(iii) Write on each answer sheet (1) your identification code (not your name), (2) the problem number.

(iv) Write only on one side of each paper.

(v) Read each problem carefully before you start to solve it.

(vi) Hand in each problem as a unit. Number your pages. Turn in your formula sheet with your solutions. Make sure that your name is NOT on the formula sheet, only your ID!

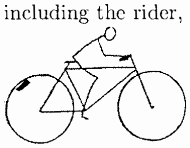
2007 Candidacy Examination
Part A: Classical Mechanics
January 9, 9 am–1 pm

General Instructions:

- *Do any four out of the five problems.*
- *Read each problem carefully before beginning.*
- *Put the problem number on each page.*
- *Put your I.D. code on the top of each page.*
- *Write only on one side of each sheet of paper.*
- *Number your pages.*
- *Write clearly and neatly.*
- *Hand in each problem as a unit. Turn in your formula sheet with your problem solutions. Make sure your name is NOT on the formula sheet, only your I.D.!*
- **IMPORTANT: Explain your approach to solving the problem before you begin. This will count 5 points. The rest of the problem will count 95 points.**

Problem 1 In Hohmann transfer between circular orbits of radii a_1 and a_2 (with $a_2 > a_1$) around the Earth, a rocket motor is fired briefly while the spacecraft is in the a_1 orbit in order to change the orbit into an elliptical one with a low-point (perigee) at a_1 and a high-point (apogee) at a_2. Once the spacecraft has reached apogee, the rocket motor is fired briefly once again in order to circularize the orbit at a_2. Calculate the time interval between the two firings of the rocket motor.

Problem 2 Model a bicycle as having two wheels of equal mass m and radius R, with all the mass of each wheel concentrated at the rim for simplicity. Let the mass of the rest of the bicycle, including the rider, be M.



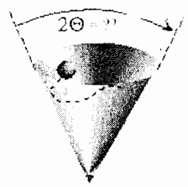
Assume that the bicycle is initially coasting along a horizontal road and that there is no friction in the wheel bearings, when a frictional force f is applied to the rim of the rear wheel by means of a brake. If the wheels do not slip, calculate (a) the deceleration of the bicycle, and (b) the magnitude and direction (forward or backward?) of the horizontal force exerted by the ground on *each* wheel. (Hint: you'll need to set up equations for the torque on each wheel, as well as the total center of mass motion. Your answers should be in terms of m, R, M and f.)

Problem 3 A solid ball of mass m and radius a rolls without slipping near the bottom of a fixed cylindrical tube of radius R. The motion is restricted to the direction perpendicular to the axis of

the tube.
(a) Find the Lagrangian for the system. Be sure to define the coordinates.
(b) Set up but do not solve Lagrange's equations
(c) Find the frequency of small oscillations.
(The moment of inertia of a solid sphere of radius a around its center is $I = \frac{2}{5}ma^2$.)

Problem 4 A long chain with mass per unit length μ lies straight on a frictionless table. A small length of chain, l_0, is made to hang straight over the edge of the table and then released.
(a) Write down and solve the equations of motion for this problem, at any time before the chain has completely left the table. What is the functional form of the velocity for this period of time?
(b) Assume now that there is some (kinetic) friction between the chain and the table. Does the form of the solutions (the velocity in particular) change qualitatively? Explain in as much detail as necessary to make your point.

Problem 5 A mass m is sliding frictionlessly inside an inverted circular cone of polar angle Θ (i.e., the cone's total opening angle is 2Θ) under the effect of gravity g.



(a) Derive the Lagrangian and Lagrange equations of motion for the radius r and the angle ϕ (r, ϕ are ordinary spherical coordinates; the origin is taken at the apex of the cone).
(b) Derive the Hamiltonian and Hamilton's equations of motion. Note any conserved quantities.
(c) Consider nearly circular orbits and estimate the radial and angular frequencies, ω_r and ω_ϕ, in terms of the angle Θ.
(d) Compute the angles $\Theta_{m:n}$ that could give closed orbits: $\omega_r/\omega_\phi = m/n$, with $m/n = 1/1, 3/2$ and $2/1$ (whichever are possible).
(e) For a cone with angle Θ slightly greater than $\Theta_{1:1}$, is aphelion precession in direction of orbit (prograde) or against it (retrograde)?

Problems for Qualifying Examination 2007 (E&M Part)

1. The time averaged potential of a neutral hydrogen atom is given by

$$\phi(\vec{r}) = \frac{q}{4\pi\epsilon_o r} e^{-2r/a_o}(1 + r/a_o)$$

where q is the magnitude of the electronic charge and a_o is the Bohr radius. Find the charge distribution which produces this potential and explain your results from the physics point of view.

2. Experimentally it is well known that low frequency electromagnetic fields do not penetrate too deep into normal metals (good conductors). On the other hand at sufficiently high frequencies metals become transparent to electromagnetic radiation. Explain why by

a) Deriving expression for the frequency dependent dielectric constant $\epsilon(\omega)$ within Lorentz-Drude model.

b) Derive high- and low- frequency limits for $\epsilon(\omega)$.

c) Estimate a magnitude of the electromagnetic radiation penetration depth (so called "skin depth") in copper at 50 Hz ($N_{Cu} \sim 8 \times 10^{28}$ atoms/m^3).

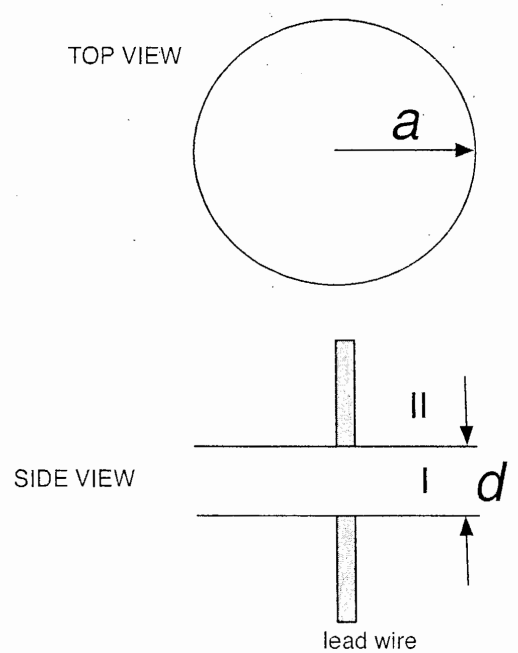
3. A point particle of charge q and mass m is placed at a distance D from an infinite grounded conductive plane. Find the induced charge distribution on the conducting plane, force experienced by the charge q due to the induced charge and energy stored in the system. If the charge q is released from the rest, how long will it take for the charge to hit the plane.

4. A parallel plate capacitor is made of circular plates as shown in Figure below. The voltage across the plates (supplied by long, resistance-less lead wires as illustrated) has the same time dependence $V(t) = V_o \cos(\omega t)$. Assume $d << a << c/\omega$ so that fringing of the electric field and retardation can be ignored.

a) Use Maxwell equations and symmetry arguments to determine the electric and magnetic fields in region I as a function of time.

b) What current flows in the lead wires and what is the current density in the plates as a function of time?

c) What is the magnetic field in region II? Relate the discontinuity of $\vec{B}$ across a plate to the surface current in the plate.



5. Consider an electromagnetic wave given by

$$\vec{E}(x,y,z,t) = \vec{E}_o e^{i(kz-\omega t)}, \ \vec{B}(x,y,z,t) = \vec{B}_o e^{i(kz-\omega t)}$$

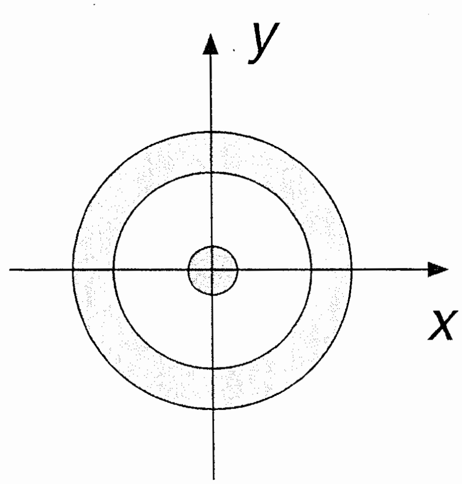
where $\vec{E}_o(x,y)$ and $\vec{B}_o(x,y)$ lie in the x-y plane.

a) Use Maxwell equations to find relationship between k and ω and the relationship between $\vec{E}_o(x,y)$ and $\vec{B}_o(x,y)$. Show that $\vec{E}_o(x,y)$ and $\vec{B}_o(x,y)$ satisfy the equations for electrostatics and magnetostatics in the free space.

b) What are the boundary conditions for $\vec{E}$ and $\vec{B}$ on the surface of a perfect conductor?

c) Let a wave like the one given above propagate along a transmission line like that shown in the figure below. Assume the central cylinder and the outer sheath are perfect conductors. Sketch the electromagnetic field pattern for a particular cross-section. Indicate the signs of the charges and the current directions in the conductors.

d) Derive expressions for $\vec{E}$ and $\vec{B}$ in terms of the charge per unit length, λ, and the current, i, in the central conductor.



2007 Ph.D. Candidacy examination
Quantum Mechanics
Jan. 11, 2007, 9:00AM-1:00PM

General instructions:

(i) Please do four problems of your choice. No more, no less.

(ii) Explain briefly your approach before solving each problem (to which 5 points will be given).

(iii) Write on each answer sheet (1) your identification code (not your name), (2) the problem number.

(iv) Write only on one side of each paper.

(v) Read each problem carefully before you start to solve it.

(vi) Hand in each problem as a unit. Number your pages. Turn in your formula sheet with your solutions. Make sure that your name is NOT on the formula sheet, only your ID!

Problem 1. Consider a one-dimensional single particle system with Hamiltonian $H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + \frac{1}{2}\gamma(\hat{x}\hat{p} + \hat{p}\hat{x})$, where γ is small. Calculate the lowest order non-zero correction to the energy of an arbitrary state $|n>$ of the unperturbed quantum oscillator.
(You may want to use the following equations, $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^+)$, and $\hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a} - \hat{a}^+)$. $\hat{a}$ is the lowering operator as $\hat{a}|n> = \sqrt{n}|n-1>$, and $\hat{a}^+$ is the raising operator as $\hat{a}^+|n> = \sqrt{n+1}|n+1>$.)

Problem 2. Consider a system whose initial state $|\Psi(0)\rangle$ and Hamiltonian are given by

$$|\Psi(0)> = \frac{1}{5}\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \quad \mathrm{H} = \hbar \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & 5 & 0 \end{pmatrix}$$

(a) If a measurement of the energy is carried out, what values would be obtained and with what probabilities? (60pts)

(b) Find the state of the system at a later time t. $|\Psi(0)\rangle$ needs to be expanded in terms of the eigenvectors of H. (20pts)

(c) Find the average total energy of the system at time $t = 0$ and later time t. Are these values different? (20pts)

Problem 3. A body has a magnetic moment $\mathbf{M} = \mu\mathbf{J}$, where $\mathbf{J}$ is the angular momentum and μ is a constant. The energy of this body in magnetic field $\mathbf{B}$ is given by the Hamiltonian $H = \frac{\mathbf{J}^2}{2I} - \mathbf{M}\cdot\mathbf{B}$, where the I constant is the moment of inertia. For convenience, choose the z-axis along the direction of $\mathbf{B}$ so that $\mathbf{J}\cdot\mathbf{B} = J_z B$.

(a) What is the energy spectrum or eigenvalues of H? (20pts)

(b) What operators can be diagonalized simultaneously with H and why? (20pts)

(c) Write out the equation that $\frac{dJ_x(t)}{dt}$ satisfies in the Heisenberg picture. (30pts)

(d) Given $J_x(0)$ and $J_y(0)$ at the initial moment $t = 0$, determine $J_x(t)$ at arbitrary moment t in the Heisenberg picture. (30pts)

Problem 4. Consider a diatomic molecule with fixed interatomic separation (i.e., rigid rotator) that is constrained to rotate in the xy plane. Suppose that its moment of inertia about an axis normal to the xy plane and passing through its center of mass is $I = \mu R^2$, where μ is the reduced mass and R is the interatomic separation.

(a) Find eigenvalues and normalized eigenfunctions of the rotator. (40pts)

(b) Suppose that the molecule has an electric-dipole moment $\vec{p}$ along the internuclear axis, and a uniform electric field $\vec{\varepsilon}$ is applied to the molecule. The electric field lies in the xy plane. Using the perturbation method, find the first non-vanishing term in the modification to the energy of the rotator. (60pts)

[Hint: The Hamiltonian is H= $\frac{\hat{P}^2}{2\mu}$, and the Laplacian in spherical polar coordinates is $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$. Or, you may write the Hamiltonian as H= $\frac{\hat{L}_z^2}{2I}$ where L_z is the z-direction angular momentum.]

Problem 5. Let $\mathbf{S}_i$ (i=1,2) denote the spin vectors of two spin-$\frac{1}{2}$ particles. The interaction between the two spins is given by $H = V_0(\mathbf{S}_1 \cdot \mathbf{S}_2 - 3S_{1z}S_{2z})$. Find the energy eigenvalues and eigenstates using the product basis $|s_1 m_1> \otimes |s_2 m_2>$ of two spin momenta.