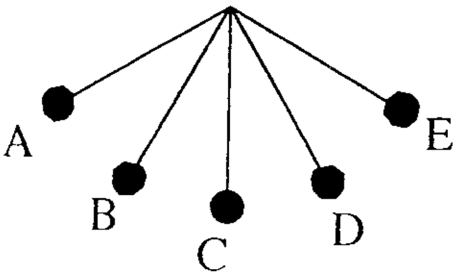
CANDIDACY EXAM, 1999

A 1

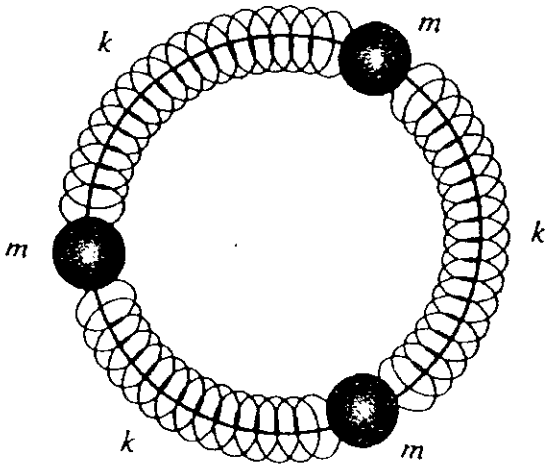
A pendulum bob of mass m on a string of length L is pulled back through an angle θ_0 and released from rest. Consider the motion at points A, B, C, D, E shown, where A and E are the high points, C is the low point, and B and D are the points where $\theta = \pm\theta_0/2$.



(a) At each point, specify whether the acceleration $\vec{a}$ is zero or nonzero, and *qualitatively* describe (either in words or with a drawing) its approximate direction if it is non-zero.
(b) Find the equation of motion for the angular displacement θ.
(c) Find $\vec{a}$ quantitatively, at points A, B, and C, giving your answer in terms of m, L, and θ_0.

A 2

Three masses, each of mass m, are interconnected by identical massless springs of spring constant k and are placed on a smooth circular hoop as shown in the figure

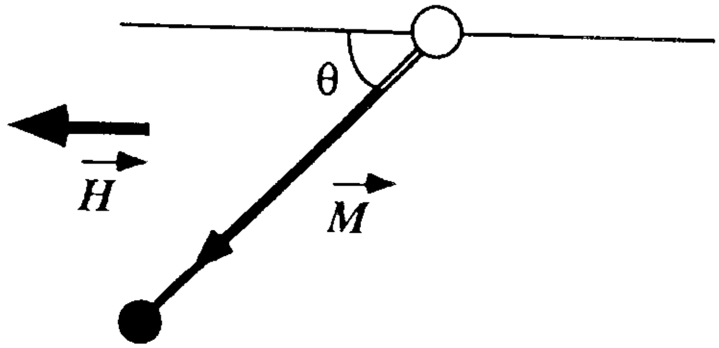


The hoop is fixed in space. Neglect gravity and friction.

Determine the natural frequencies of the system, and the shape of the associated modes of vibration.

A 3

Consider a massless, rigid rod pendulum of length l pivoted about a frictionless axle; the axis of rotation is fixed. There is a small sphere of mass m attached to the end of the pendulum (see Fig.). The pendulum rod, though massless, has a magnetization $\boldsymbol{M}$, which points down along the length of the rod; this magnetization is rigidly connected to the rod. In addition to gravity, the pendulum is subjected to a uniform magnetic field of magnitude $\boldsymbol{H}$ which points horizontally in the plane of the pendulum motion. Note that the magnetic part of the potential energy is $-\boldsymbol{M}\cdot\boldsymbol{H}$.



(a.) Write down the Lagrangian for this system and derive the equation(s) of motion

(b.) In the limit of small oscillations, find the general form of the dependence of the angle θ on time. Be sure to give a specific form for the frequency of oscillation.

B1

The space between two concentric spheres of radii R_1 and R_2 ($R_1 < R_2$) is charged with a charge density given by $\rho = \alpha/r^2$.

(a) Find the total charge q

(b) Find the potential Φ in regions $r < R_1$, $R_1 < r < R_2$, and $r > R_2$;

(c) Find the electric field strength in regions $r < R_1$, $R_1 < r < R_2$, and $r > R_2$.

B2

A point charge q is located at a distance d from the center of a grounded conducting sphere of radius $a < d$.

(a) Find the electrical force on q for $d = a + \delta$, where $\delta \ll a$. Comment on the dependence of the force on δ.

(b) What is the force on the charge if $d \gg a$? Comment on the distance dependence of the force.

B3

A point charge q is placed at $x = y = b$, $z = 0$. The half-plane $x = 0$, $y > 0$ is held at potential $\Phi = (y^2 + z^2)^{-1}$, and the half-plane $y = 0$, $x > 0$ is held at potential $\Phi = (x^2 + z^2)^{-1}$.

(a) Find the Green's function $G(\vec{x}, \vec{x}')$ for the region $x > 0$, $y > 0$.

(b) Find the potential $\Phi(\vec{x})$ in the region $z > 0$, $y > 0$, using your Green's function. You may leave the surface-integral term as an explicitly-formulated integral, without integrating it.

Given:

$$\Phi(\vec{x}) = \int_V \rho(\vec{x}')G(\vec{x}, \vec{x}')\, d^3x' - \frac{1}{4\pi}\oint_S \Phi(\vec{x}')\frac{\partial G}{\partial n'}\, da'$$

C1

A classical system of N distinguishable noninteracting particles of mass m is placed in a three-dimensional harmonic well:

$$U(r) = \frac{x^2 + y^2 + z^2}{2V^{2/3}}$$

a) Find the partition function and the Helmholtz free energy.

b) Regarding V as an external parameter, find the thermodynamic force P conjugate to this parameter, exerted by the system; find the equation of state and compare it to that of a gas in a container with rigid walls.

c) Find the entropy, internal energy, and total heat capacity at constant volume.

Given: $\int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}$, $\quad \int_0^\infty x^2 e^{-a^2x^2}dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$ $\quad \frac{1}{4a^3}\sqrt{\pi}$

C2

Consider a system of N identical, non-interacting, one-dimensional, simple harmonic oscillators, each with mass m and frequency ω, located at fixed positions x_i, $i = 1, \ldots, N$

(a) Find the classical partition function Z_c

(b) Using Z_c find the classical energy equation of state. [Hint: You should already know the answer!]

(c) Find the quantum partition function Z_q

(d) Use Z_q to find the quantum energy equation of state.

(e) Show that in an appropriate limit the results of (b) and (d) agree. Describe this limit.

(f) Which of the three types of statistics (Boltzmann, Bose or Fermi) does the system obey? Justify your answer.

Note: You may find the following formulas useful:

$$\int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}, \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (0 < x < 1)$$

C3

Superparamagnetism is a phenomenon in which the total magnetic moment μ of a small ferromagnetic particle acts as a single, thermalized unit, much like a *classical* spin.

Consider a system of N such particles (noninteracting) subjected to a magnetic field of magnitude H along the z direction.

(a.) Calculate the classical partition function for this system of moments.

(b.) Show that the overall magnetic moment m for this system is of the form

$$m = N\mu\left(\coth(\beta\mu H) - \frac{1}{\beta\mu H}\right)$$

(Hint: Integration by parts may prove useful)

D1

An excited state wave function for a one electron atom is

$$Y_{ex} = C\,(r/a_o)\exp[-r/(2a_o)]\cos(q),$$

where C is a constant and a_o is the Bohr radius.

(a) Normalize the wave function.

(b) Find the most probable value for the distance the electron is from the nucleus.

(c) Find the expected value for the distance the electron is from the nucleus.

(d) Find the probability that the electron lives in the "arctic polar region" of the atom, i.e., the electron is inside the conical region within 23.5° of the z-axis.

(e) Find the expected value of the electrostatic potential energy of the electron.

Remember: $\int_0^\infty x^n \exp(-ax)\,dx = n!/a^{n+1}$, and $\int \sin(ax)\cos^m(ax)\,dx = -\cos^{m+1}(ax)/[(m+1)a]$

D2

Consider a square lattice of 2N ions of alternating charge $\pm e/3$ separated by a distance R from each other. Model the Pauli repulsion as A/R^n between nearest neighbors.

(a) Find the total potential energy of an ion considering the first, second, third, and fourth nearest neighbor ions.

(b) If at equilibrium the nearest neighbor separation is 0.1nm find A.

(c) If the energy required to remove one ion pair is 8eV, find n.

(d) If an ion is slightly displaced from its equilibrium position, determine the effective spring constant which acts to restore the ion to its original position.

D3

A supernova exploded in the Large Magellanic Cloud, which is about 50 Kpc away (1.5×10^{21} m). A neutrino detector in Japan registered a burst of detections spread out over several seconds due to this event. The earliest detected neutrinos were about 1.0 Mev more energetic than those arriving 3.0 seconds later. Assume all of the neutrinos were emitted simultaneously from the center of the supernova.

(a) Explain why the neutrino detections imply a non-zero rest-mass for the neutrino.

(b) Estimate the velocity difference between the earliest detected neutrinos and those arriving 3.0 sec later.

(c) Estimate the relativistic factor γ for the neutrinos.

(d) Estimate the rest-mass of the neutrino in eV.

D4

Consider a nucleus of spin J = 3/2 which is subjected to a pure quadrupole interaction of the form

$$H = q(J_x^2 + J_y^2)$$

(a) Find the energy levels. What is the degeneracy of the each level?

(b) Express the eigenfunctions of the nucleus in terms of the eigenstates of the magnetic substates.

Note: $J_+|j,m\rangle = [j(j+1)-m(m+1)]^{1/2}|j,m+1\rangle$, $J_-|j,m\rangle = [j(j+1)-m(m-1)]^{1/2}|j,m-1\rangle$

E1

Consider s-wave ($l = 0$) scattering of particles by a spherically symmetric short range potential

$$V(r) = -V_0 \qquad r < r_0.$$
$$0 \qquad r > r_0.$$

The radial part of the solution $R(r) = rU(r)$ satisfies

$$\left[\frac{d^2}{dr^2} + \frac{2m}{\hbar^2}(E - V)\right] U(r) = 0$$

and has the form

$$U_{in}(r) = A\sin(Kr)$$

$$U_{out}(r) = B\sin(kr + \delta_0)$$

where A, B are some constants.

(i) Determine K, k and δ_0.

(ii) From the expression for δ_0, discuss how the scattering of slow particles, $E \ll |V_0|$, allows you to determine whether the potential V is attractive ($V_0 > 0$) or repulsive ($V_0 < 0$).

You may find the following expansions useful:

$$\tan x = x + \frac{x^3}{3} + \ldots$$

$$\tanh x = x - \frac{x^3}{3} + \ldots$$

E2

A particle is in a one-dimensional "box" of length L. Imagine that, initially, there is a barrier (infinitely high in energy) that keeps the particle confined to the left half of the box, $0 < x < L/2$.

(a) If the particle's mass is m, what is the energy and (normalized) wavefunction for the minimum energy state when it is confined to the left half of the box? Assume that this lowest energy state is the initial state of the particle at $t < 0$.

(b) Imagine that, at $t = 0$, the wall is instantaneously removed. Find the energies E_n and (normalized) eigenfunctions $\psi_n(x)$ allowed for the particle in the whole box.

(c) If the initial state of the particle is the one you calculated in (a), write down the wavefunction at any later time, after the barrier has been removed.

(d) Use the result in part (c) to write down an expression for the average energy of the particle. How does this compare to (a)? Is the particle's energy well-defined anymore, or not? If not, what is the energy spread?

You may find the following result useful:

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{n^2}{(n^2-4)^2} = \frac{\pi^2}{16}$$

E3

The Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

describes a one-dimensional simple harmonic oscillator.

(a) Using the Heisenberg picture, set up the equations of motion for this system.

(b) Find the general solution for $x(t)$ and $p(t)$ in terms of the operators $x(0)$, $p(0)$.

(c) Find the commutation relations $[x(t_1), x(t_2)]$, $[p(t_1), p(t_2)]$, and $[x(t_1), p(t_2)]$ for the oscillator of the preceding part, where the symbols are Heisenberg picture operators. Verify these results for the special case $t_1 = t_2$.