INTEREST, PROFITS, AND SUBOPTIMALITY IN A DEMAND-CONSTRAINED MACRO MODEL*

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Keynes argues that under pessimistic expectations, investment demand falls short of full-employment savings. Consequently, income adjusts to bring savings into line with investment, resulting in unemployment and/or excess capacity.

Classical theory, as reconstituted in postwar textbooks, counters that investment and full-employment savings are equilibrated by adjustment of the interest rate. Thus, current output will obtain its optimal level, at full employment and full capacity, despite pessimistic expectations of future demand. The only way this can fail is if some type of wage or price rigidity prevents the stock of real money balances from accommodating the demand for them at the full-employment interest rate.

Fig. 1 illustrates the Keynesian and classical views in a traditional loanable funds diagram. Here, the full-employment supply of savings is given by $S^*$, investment demand under optimistic expectations is $I^*$, and the full-employment interest factor (unity plus the interest rate) is $r^*$. Pessimistic expectations shift investment demand to $I'$, reducing the full-employment interest factor to $r'$ in the classical model; or reducing employment, income, and, hence, savings to $S'$ in the simplest Keynesian model.

General equilibrium theory provides a third view on interest rate determination. Here, competition over prospective profits leads the interest rate to adjust to the return on capital (adjusted for risk), so prospective profits are eliminated, provided capital is mobile.

To reconcile this view with either the Keynesian or classical view, depicted in Fig. 1, it should first be noted that the general equilibrium model of interest rates is a multi-period model, while the textbook models of Fig. 1 are not. They are one-period models, with views of the future imprecisely summarised in the investment demand function. Furthermore, in those models, views of the future are not analysed for prospective profits, or their consequences. Graphically, the inframarginal triangle between the investment demand curve and the interest rate line is not analysed or explained. How can such prospective profits survive competitive activities?

The present paper proposes to examine this issue in as simple a model as possible. The key simplifications are as follows: First, the simplest multi-period

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model which can adequately analyse investment, prospective profits, and the relationship between the two, is a two-period model. This will also help us to distinguish between current (ex post) profits, which have no behavioural implications (since the current capital stock is predetermined), and prospective (ex ante) profits, which stimulate a competitive response. The length of the period under consideration is the time required for a competitive response to prospective profits.

Second, we adopt a choice-theoretic variant of the Malinvaud (1980, 1982, 1983) quantity-constrained model, where firms' current investment demand is depressed by pessimistic expectations of future quantity constraints on the demand for output. Other simplifications include the abstraction from risk and liquidity preference.

The substance of the paper is as follows: Section I develops the basic two-period model, where pessimistic firms face a binding constraint on future demand (and, possibly, an irreversibility constraint on investment). A constrained social optimum is derived. Section II shows that the classical interest rate adjustment, which secures full employment and full capacity for the current period, achieves the constrained optimum. But in so doing, it is shown, factor prices must fall below their expected marginal productivities (unless the irreversibility constraint binds), and, as a result, prospective profits appear.

The next three sections analyse three competitive responses to prospective profits: (1) entry; (2) expenditures in pursuit of market share; and (3) arbitrage on the capital markets. Each of these three responses can dissipate the prospective profits which gave rise to them, but only by introducing inefficiencies. These are, respectively: (1) sacrifice of scale economies; (2) 'Directly Unproductive Profit-Seeking (DUP) Activities', to use the term coined by Bhagwati (1982); and (3) high interest rates, which depress investment, and idle current resources in the Keynesian fashion depicted in Fig. 1. Thus, the competitive pursuit of profits in a demand-constrained system leads to suboptimal configurations of one type or another (or some combination).

Perhaps the most interesting perspective which emerges from this analysis is that Keynesian unemployment and/or excess capacity can be understood as
simply one of (at least) three types of inefficiencies which prevent profits from arising in situations of pessimistic expectations. These suboptimalities, it should be emphasised, are relative to the second-best optimum, constrained by pessimistic expectations of future demand.

An appendix provides comparative static analyses of all the models in the paper with respect to expectations of future demand, under some simplifying assumptions, such as the imposition of a CES utility function. The optimality results, however, do not depend on these assumptions.

I. THE CONSTRAINED SOCIAL OPTIMUM

Consider a simple two-period model. The utility function of a representative consumer is $U(C_o, L_o, C_i, L_i)$, where $C$ is consumption, $L$ is labour, and subscripts denote period 0 or 1. Production is carried out under the same technology, $F(K, L)$, in each period. There is a predetermined capital stock, $K_o$, and period 0 output can be used for consumption or gross investment, $I$. A fraction $\delta$ of $K_o$ survives to period 1, so the period 1 capital stock, $K_1$, is $\delta K_o$ plus the gross investment of period 0. Gross investment, of course, must be financed out of savings:

$$K_1 - \delta K_o \equiv I = S \equiv F(K_0, L_0) - C_0. \quad (1)$$

Also, investment is irreversible, so

$$K_1 - \delta K_o \equiv I \geq 0.$$

In period 1, all output is consumed, since there is no period 2:

$$F(K_1, L_1) = C_1. \quad (2)$$

In restricting $F$, we recall the product exhaustion debate, which established that marginal productivity payments result in positive or negative profits as $F$ observes decreasing or increasing returns to scale. Hence, for Walrasian zero-profit (free entry) equilibrium to exist, $F$ must observe constant returns over at least some portion of its domain. Classical macro models, then, typically simplify matters by assuming globally constant returns, unless the indeterminacy of individual firm size is a concern, in which case $F$ is assumed to be dual to a U-shaped cost curve. In this Section, and Section II, we adopt the global CRS assumption, as usual, and then consider U-shaped cost curves in Section III. Of course, our focus is not Walrasian equilibrium, with marginal productivity payments, but demand-constrained equilibrium, where marginal productivity payments will not hold, so we will find a different relationship between returns to scale and profits.

Now we assume firms perceive an aggregate demand constraint$^{1}$ for period 1,

$$C_1 \leq \bar{C}.$$

$^1$ Antecedents here include Malinvaud (1980, 1982, 1983), Nickell (1978, pp. 72–5), and Neary and Stiglitz (1983). In a multi-period model, where current demand depends on expectations of future demand (through investment), terminal period demand expectations must be fixed to close the model, as in all intertemporally dependent rational expectations models. Blanchard and Kahn (1980, pp. 1306–7) point out the similarity between multiplier-accelerator models and asset market models with rational bubbles. See also Blanchard (1982, pp. 3–6) for a brief overview of the non-uniqueness problem in macroeconomics.
We can evaluate welfare today, assuming these expectations of the future turn out to be correct (or self-fulfilling). Consider the social optimum by forming the Lagrangian

\[ U(C_0, L_0, C_1, L_1) + \mu_0[F(K_0, L_0) - C_0 - (K_1 - \delta K_0)] + \mu_1[F(K_1, L_1) - C_1] + \mu_2(C - C_1) + \mu_3(K_1 - \delta K_0). \]

The optimal configuration for \( C_0, L_0, C_1, L_1, K_1, \mu_2, \) and \( \mu_3 \) is given by (1)-(2), and

\[ -\frac{U_{L_0}}{U_C} = F_{L_0}, \quad \mu_2 = 0; \quad \text{or} \quad C_1 = C, \quad \mu_2 = \frac{U_{C_1} + U_{L_1}}{F_{L_1}} > 0, \]

\[ -\frac{U_{L_1}}{U_C} = \frac{F_{L_1}}{F_{K_1}}, \quad \mu_3 = 0; \quad \text{or} \quad K_1 = \delta K_0, \quad \mu_3 = \frac{U_{C_0} + F_{K_1}}{F_{L_1}} > 0, \]

where subscripts on \( U \) and \( F \) denote partial derivatives.

In the case where neither constraint binds (\( \mu_2 = \mu_3 = 0 \)), all marginal rates of substitution among current and future leisure and consumption should equal the corresponding marginal rates of transformation. If the demand constraint binds (\( \mu_2 > 0 \)), then \( C_1 = \bar{C} \), and the link is severed between MRS's and MRT's involving \( C_1 \). Similarly, if the irreversibility constraint binds investment (\( \mu_3 > 0 \)), then \( K_1 = \delta K_0 \) and the link is severed between MRS's and MRT's involving future and present variables, since investment is what links the present and the future.

There are four possible cases, depending on which constraints, if any, bind. Of these, perhaps the least relevant case is the one where the irreversibility constraint binds, but the demand constraint does not. That is, it seems hard to imagine why an economy would like to disinvest, unless there is a demand-induced recession. Accordingly, we shall not consider this case any further.

Instead, we shall assume that if \( \bar{C} \) is large, neither constraint binds; as \( \bar{C} \) falls below the Walrasian level of \( C_1 \), the demand constraint binds; and finally, as it falls further, demand-constrained investment falls below the irreversibility floor. The range we shall focus on most is the intermediate range, where the demand constraint binds, but not the irreversibility constraint. This would seem to cover a rather wide range, including not only 'growth' recessions, but also recessions where the typical firm is undertaking at least some replacement investment.

II. THE CLASSICAL SOLUTION IS OPTIMAL, BUT GENERATES PROFITS

The previous section's solution to the constrained optimum was described without reference to market prices. In this Section we will show that the classical model of the determination of factor prices – the interest rate in particular – solves for the constrained optimum. However, we will also show that this solution generates profits over an important range of cases.

According to the classical model, the consumer recognises no constraints other than the budget constraint – specifically, no involuntary unemployment. Hence,
the consumer's labour supplies, consumption, and savings decisions satisfy the usual marginal conditions:

\[-U_{L0}/U_{c0} = w_0, \quad (6)\]
\[-U_{L1}/U_{c1} = w_1, \quad (7)\]
\[U_{c0}/U_{c1} = r_1, \quad (8)\]

where \(w_i\) is the real wage in period \(i\). Here, \(r_1\) is unity plus the real interest rate prevailing in period \(1\), although established in period \(0\); it is the interest factor appearing in the loanable funds diagrams.

Let us now turn to the firms. By the simplifying assumption of CRS (to be relaxed in Section III), the number of firms is indeterminate, but also irrelevant. We can, therefore, legitimately analyse the economy 'as if' there were only one competitive firm. This causes no loss of generality, and considerable gain in ease of notation.

Profits in period \(0\) are

\[\Pi_0 = F(K_0, L_0) - w_0 L_0 - (r_0 - \delta) K_0.\]

Here, \(r_0\) is the interest factor prevailing in period \(0\), but predetermined from period \(-1\). Hence, the service costs of capital in period \(0\) are all predetermined, comprising interest, \((r_0 - 1)K_0\), and depreciation, \((1 - \delta)K_0\). Maximising with respect to the single decision variable, \(L_0\), the classical firm takes prices as parametric. If the classical interest rate adjustment holds, then firms will face no binding constraint on current sales, so we get the usual current labour demand,

\[F_{L0} = w_0. \quad (9)\]

The firm's investment and future labour demand are found by maximising prospective profits,

\[\Pi_1 = F(K_1, L_1) - w_1 L_1 - r_1 K_1, \]

with respect to both \(L_1\) and \(K_1\), subject to the relevant constraints, to be discussed.

The capital costs, \(r_1K_1\), represent interest and \(100\%\) depreciation, since there is no period 2.

Now what constraints does the firm recognise in making its investment and future employment decisions? First, of course, it recognises the irreversibility constraint. Second, we want the classical firm's investment demand to allow a role for pessimism, as depicted in Fig. 1, so we assume the firm does recognise the quantity constraint on future demand. The asymmetric treatment of current and future demand constraints is a device for exploring the classical contention that interest rates can adjust to pessimistic expectations in a way that achieves the constrained social optimum. (In Section V, we let future pessimism spill over to the present in Keynesian fashion.)

Formally, then, the classical firm is here interpreted to maximise the Lagrangian

\[\Pi_1 + \lambda[C - F(K_1, L_1)] + \phi(K_1 - \delta K_0),\]
The first-order conditions are given by

\[ w_1 = F_{L_1}, \quad \lambda = 0; \quad \text{or} \quad F(K_1, L_1) = \tilde{C}, \quad \lambda = (F_{L_1} - w_1)/F_{L_1} > 0, \]

\[ w_1/r_1 = F_{L_1}/F_{K_1}, \quad \phi = 0; \quad \text{or} \quad K_1 = \delta K_0, \quad \phi = r_1 - w_1 F_{K_1}/F_{L_1} > 0. \]

The system (1)–(2), (6)–(11) suffices to solve for the five quantities \((C_0, L_0, C_1, L_1, K_1)\) and the three factor prices \((w_0, w_1, r_1)\).

Substituting (6)–(8) into (9)–(11) for the factor prices, it is readily verified that the optimum conditions (3)–(5) are satisfied. The firm’s multipliers, \(\lambda\) and \(\phi\), accurately reflect the planner’s multipliers, \(\mu_2\) and \(\mu_3\), up to a multiplicative constant, \(1/U_{C_1}\). Hence, we have shown that the classical interest rate adjustment to pessimism, as presently interpreted, does indeed reproduce the constrained social optimum.

However, this proposed solution has implications for prospective profits which are not commonly appreciated. To see this, rewrite (10) and (11) as

\[ w_1 = (1 - \lambda) F_{L_1}, \]

\[ r_1 = (1 - \lambda) F_{K_1} + \phi. \]

Substituting into the expression for \(\Pi_1\), we get

\[ \Pi_1 = F(K, L) - wL - rK \]

\[ = LF_L + KF_K - (1 - \lambda) LF_L - (1 - \lambda) KF_K - \phi K \]

\[ = \lambda LF_L + KF_K - \phi K \]

\[ = \lambda \tilde{C} - \phi \delta K_0, \]

where all variables pertain to period 1, except where noted. Let us consider the three ranges of \(\tilde{C}\) discussed at the end of Section I.

Range (i): \(\tilde{C}\) is so large that neither constraint binds, \(\lambda = \phi = 0\). Then we have the Walrasian solution, with marginal productivity payments, and there are no prospective profits, as is well-known for CRS. It may be worth some effort to illustrate this point by formally deriving the investment demand curve and examining it in a loanable funds diagram, Fig. 2, which will also illustrate Ranges (ii) and (iii) below.

The investment demand curve is given by the marginal efficiency of investment. In a discrete time model such as this, the marginal efficiency of investment curve is simply a horizontal displacement of the marginal efficiency of capital curve, since \(I = K_1 - \delta K_0\), and \(\delta K_0\) is predetermined. The marginal efficiency of capital is defined as

\[ m(K; w) = \frac{d}{dK} \left[ \max_L \{F(K, L) - wL\} \right] \quad \text{such that} \quad F(K, L) \leq \tilde{C}. \]

Over the range under consideration, the demand constraint does not bind, so the envelope theorem gives us

\[ m(K; w) = F_K(K, L)/F_L(K, L) = w, \quad F(K, L) < \tilde{C}. \]

As Malinvaud (1983) notes, parametric prices and quantity constraints mean the firm believes it faces a kinked demand curve of the type discussed by Negishi (1979) and Hahn (1978, 1980).
Since $F$ is linear homogeneous, the wage determines the capital–labour ratio, which, in turn, determines the marginal productivity of capital. Therefore, the marginal efficiency of capital is independent of $K$ over this range, and equals the average return to capital, $(F - wL)/K$. In Fig. 2, the full-employment savings curve, $S^*$, intersects the investment demand curve in this range, $AB$, so the cost of capital, $r^*$, equals both $m$ and the average return to capital. Consequently there is no inframarginal area between the $m$ curve and the interest rate line: prospective profits are zero.

![Diagram](image)

Fig. 2. Marginal efficiency of investment, classical interest rate, and profits.

Range (ii): $\bar{C}$ falls below the Walrasian level, so the demand constraint binds, $\lambda > 0$, but the irreversibility constraint does not, $\phi = 0$. Here, the perfectly elastic portion of the investment demand curve extends only to $B'$, at which point the demand constraint binds. Beyond this point, the marginal efficiency of investment, (13), is given by

$$m(K; w) = (1 - \lambda) \frac{F_K}{F_L} | F(K, L) = \bar{C},$$

i.e. the wage times the marginal rate of technical substitution. It represents the reduction in labour costs from substituting capital, while meeting the fixed demand. Obviously, it declines with $K$, by convexity, as we move along the $\bar{C}$-isocuant. Note that as we move along the investment demand curve, $L_1$ increases with $K_1$ over the unconstrained segment $AB'$, then decreases over the constrained segment $B'C'$.\(^1\)

When $\bar{C}$ is in Range (ii), the full-employment savings curve intersects the investment demand curve on its constrained segment, say at point $C'$, so the classical interest rate is $r'$ and investment is $I'$. The area under the investment demand, or marginal efficiency curve, $AB'C'$ equals $\bar{C} - wL$, so prospective profits, $\bar{C} - wL - rK$, are given by the area of the trapezoid $AB'C'D'$, which equals $\lambda \bar{C}$, from (12). The diagram is essentially the same as the textbook

\(^1\) The kink in the investment demand curve is smoothed out when risk is introduced, as in Costrell (1983).

\(^2\) The profit-maximising variations in $L$ as we move along the investment demand curve contrast with the dubious constructions of some writers, e.g. Branson (1972, pp. 208–11). In these constructions $L$ is held constant at some unspecified level. Therefore, either that $L$ is not maximising profits, given $w$, or else $w$ is presumed to vary as we move along the investment demand curve. Either one violates the assumption of price-taking, profit-maximising firms.
diagram 1 a, with the important difference that our formal analysis has shown
the curve to be perfectly elastic over the region where the demand constraint
does not bind. When the constraint does bind, an inframarginal area (it is a
trapezoid, rather than a triangle) materialises, because the classical interest rate
must fall below the marginal productivity of capital to stimulate sufficient
investment to absorb full-employment savings.1

Range (iii): $C$ falls so low that in addition to the demand constraint, the
irreversibility constraint binds as well, $\lambda, \phi > 0$. Here, the marginal efficiency
curve shifts to $AB'C'E'$, intersecting the full-employment savings curve at
negative gross investment. The irreversibility constraint forces the firm to invest
nothing, so the full-employment interest rate is $r'$. Here, the trapezoid $AB'C'D'$
again equals $\lambda \bar{C}$, what profits would be if negative investment were allowed at that
interest rate. However, the firm’s profits are reduced by the area of the triangle
$C'r'E'$, which equals $\phi \delta K_0$, the loss from carrying more capital than the firm
would like (i.e. the excess of the extra interest cost over the further reduction in
labour costs). As depicted, prospective profits are still positive, $\lambda \bar{C} > \phi \delta K_0$, but
with further reductions in $\bar{C}$, they would go negative.

The relationship between $\bar{C}$ and prospective profits, covering the three ranges
discussed, is summarised in Fig. 3. As we leave Range (i), the demand constraint
binds, so $\lambda$ becomes positive and profits appear. They may reach a maximum in
Range (ii), as depicted, before the irreversibility constraint binds, since the
reduction in $\bar{C}$ may offset the rise in $\lambda$ within this range. Still, profits certainly
remain positive throughout Range (ii). It is only after we get into Range (iii),
where the irreversibility constraint binds, that the fixed costs of carrying un-
wanted capital eventually drive prospective profits negative.

This analysis may appear somewhat paradoxical, since entrepreneurial
pessimism is typically identified with losses, not profits. Indeed, we view this
paradox as a serious problem with the classical model, and its result of con-
strained optimality. This model generates prospective profits because it assumes

1 Actually, Figs. 1 and 2 oversimplify the comparative statics a bit. A deterioration of expectations,
$\bar{C}$, would not only shift the marginal efficiency curve left, but the full-employment savings curve would
also shift, due to different choices of $C_0$ and $L_0$. Furthermore, the future real wage would not be
invariant with respect to $\bar{C}$, so neither would the height of the marginal efficiency curve. Still, the
geometric depiction of profits, in each configuration, is accurate, although comparisons on the same
diagram may not be.
the interest rate falls below the average return to capital, in order to secure full-employment investment. It does not examine the competitive responses typically elicited by prospective profits.

The next two sections extend the classical full-employment model to show how competitive responses to prospective profits can prevent those profits from materialising. In so doing, however, inefficiencies are introduced. Furthermore, new paradoxes present themselves, casting further doubt on the adequacy of the classical model for analysing pessimistic investment demand. Finally, in Section V, we reconsider the Keynesian model, which argues that profits do not appear because the interest rate does not adjust to maintain full employment and full capacity. Of course, Keynesian unemployment and/or excess capacity are also departures from the constrained optimum.

III. DISSIPATION OF PROSPECTIVE PROFITS BY ENTRY AND LOSS OF SCALE ECONOMIES

Competitive analysis informs us that prospective profits invite entry. Of course, if constant returns prevail globally, the number of firms is irrelevant, so entry has no real effect on the economy's aggregates. It behooves us, then, to relax the assumption of global CRS in favour of the more acceptable assumption of variable returns to scale, associated with U-shaped cost curves.

In this section we will first establish that the optimum conditions for such a model now include the proviso that each firm operate at a point of constant returns. Then we will show that the classical interest rate model only achieves that condition in the unconstrained case; if the demand constraint binds, the classical interest rate yields prospective profits at points of CRS, so entry occurs, reducing firm size, and driving firms back up their cost curves into the region of increasing returns. In this way, a zero-profit equilibrium may be established, with full employment, but at the cost of lost scale economies. To keep the analysis focused on resolving the paradox of countercyclical profits, we confine our attention to Ranges (i) and (ii) of $C$, such that the irreversibility constraint does not bind.

Formally, we have a new variable in the model, $n_1$, the number of firms investing in anticipation of period 1 production. (The number of firms producing in period $o$, $n_0$, is predetermined.) Using the representative firm construct, we evaluate the production function and its partial derivatives at $K_i/n_i, L_i/n_i, i = o, 1$. The planning problem then is to maximise

$$U(C_0, L_0, C_1, L_1) + \mu_0[n_0 F(K_0/n_0, L_0/n_0) - C_0 - (K_1 - \delta K_0)] + \mu_1[n_1 F(K_1/n_1, L_1/n_1) - C_1] + \mu_2(C - C_1)$$

with respect to $n_1$ as well as the other variables. The first-order conditions (3)–(5) (with $\mu_3 = 0$ over the ranges considered) still hold, and the first-order condition for $n_1$ is

$$n_1 F_1 = K_1 F_K + L_1 F_L,$$

the obvious condition that each firm should be large enough to operate at the bottom of the average cost curve, at the point of constant returns. Equivalently,
we may express the solution in terms of the function coefficient (see Ferguson (1971, pp. 79–83)),

\[ e(K/n, L/n) \equiv (KF_K + LF_L)/nF, \]

the sum of the output elasticities of capital and labour. Then the optimum condition is simply

\[ e(K/n, L/n) = 1, \]

provided the second-order conditions are satisfied. (To economise on notation, we let the default subscript indicate period 1.) This, it should be noted, is independent of whether or not the demand constraint binds.

Now we ask how well the classical model meets these conditions. The consumer’s marginal conditions (6)–(8) still hold. The representative firm now chooses its inputs \( L_0', L_1', \) and \( K_1' \) to maximise

\[ \Pi_0' = F(K_0', L_0') - w_0 L_0' - (r_0 - \delta) K_0' \]

and

\[ \Pi_1' = F(K_1', L_1') - w_1 L_1' - r_1 K_1', \]

subject to the demand constraint

\[ F(K_1', L_1') < C/n_1, \]

where each firm takes its market share, \( t/n_1, \) as parametric (to be relaxed in Section IV). The first-order conditions again correspond to (9)–(11) (with \( \phi = 0 \) over the ranges considered). Thus, using (6)–(8) again, we find that the optimum conditions (3)–(5) hold, so the classical model generates the optimal values for \( C_0, C_1, L_0, L_1, \) and \( K_1, \) conditional on \( n_1. \)

The number of firms is governed by the additional condition that free entry drives prospective profits to zero. Aggregate prospective profits are

\[ \Pi = nF(K/n, L/n) - wL - rK \]

\[ = nF - (1 - \lambda) (LF_L + KF_K), \]

by (10') and (11')

\[ = nF [1 - (1 - \lambda) \epsilon]. \]

Thus, the zero-profit condition is

\[ e(K/n, L/n) = 1/(1 - \lambda) \geq 1, \quad \text{as} \quad \lambda \geq 0. \]

For Range (i), where \( \tilde{C} \) is so large that the constraint does not bind, free entry generates the full Walrasian optimum, the textbook model with each firm operating at the bottom of the average cost curve.

In Range (ii), however, where \( \tilde{C} \) falls below the Walrasian level of \( C_1, \) the classical interest rate falls below the marginal product of capital \( (\lambda > 0), \) so profits are generated at any point of constant returns. Entry drives firms back up their cost curves, into the region of increasing returns. If a zero-profit equilibrium exists, it is because entry dissipates prospective profits by the inefficiency of small-scale production.

It may be instructive to illustrate the model in a loanable funds diagram. First, let us restrict the class of production functions to

\[ F(K', L') = g[G(K', L')]. \]
where \( G \) is linear homogeneous and \( g \) is S-shaped. Hence, our production function has homothetic isoquants and

\[
\varepsilon(G) = \frac{(K'F_K + L'F_L)}{F} = \frac{(K'G_K + L'G_L)}{g'} = G g'/g,
\]

the elasticity of \( g \) with respect to \( G \). This elasticity is a decreasing function of \( G \) in the region of \( \varepsilon(G^*) = 1 \), which implies \( g'' < 0 \) in that region.

The marginal efficiency of capital in the unconstrained Range (i) is given by

\[
m(K'; w) = \frac{F_K(K', L)}{w}, \quad \frac{F_L(K', L)}{w} \leq \bar{C}
\]

since \( G \) is linear homogeneous, \( G_L \) and \( G_K \) depend only on the capital–labour ratio, \( k \). Hence, the side condition can be written as

\[
w = g'[G(K', K/k)] G_L(k),
\]

which implicitly defines \( k(K'; w) \). Total differentiation gives us

\[
\frac{dk}{dK'} = \frac{(K'G_K + L'G_L)}{(L')^2 G_L g'' - K' g'(dG_L/dk)}.
\]

This derivative is positive to the right of \( g \)'s inflection point (which includes the point of constant returns, \( \varepsilon(G^*) = 1 \)), zero at the inflection point, and negative for at least some neighbourhood to the left of the inflection point. Differentiating \((14')\), we have

\[
\frac{dm(K'; w)}{dK'} = w \frac{d(G_K/G_L)}{dk} \frac{dk}{dK'},
\]

which has the opposite sign of \( dk/dK' \). Therefore, the marginal efficiency of capital has at least one positively inclined portion, rising to a point corresponding to \( g \)'s inflection point, in the region of increasing returns, and declining thereafter through the regions of constant and decreasing returns. Fig. 4a depicts \( m \) on the assumption that it has a single hump. We have also depicted the average return to capital, \( a = (F - wL')/K' \), which bears the usual relationship to its marginal curve: it crosses \( m \) at its maximum, which is the point of constant returns, \( e = 1 \).

We have drawn in the diagram full-employment savings per firm, \( S*/n \). If there are too many firms, this curve intersects \( m \) at a point like \( A \), so the classical interest rate exceeds the average return to capital, generating prospective losses. Firms exit, \( n \) falls, and \( S*/n \) shifts right. Conversely, if there are too few firms, the classical interest rate, at point \( B \), lies below \( a \), profits appear, and entry follows, shifting \( S*/n \) left. The equilibrium is stable, at point \( D \), where the optimal number of firms, \( n* \), is obtained (ignoring the integer problem).
Now consider Range (ii), where $\bar{C}$ falls below the Walrasian level of $C_1$. Then, if each firm faced the Walrasian supply of savings, $S^*/n^*$, it would find itself unable to sell the corresponding level of $C_1$. That is, the constrained portion of the marginal efficiency curve, given by (15), becomes relevant. As Fig. 4b shows, point $D$ now lies above the constrained portion of $m$. Holding $n = n^*$ for the moment, we see that the classical interest rate, $r'$, lies below the average return to capital, as drawn, so prospective profits appear. This invites entry, so $n$ rises. This, in turn, has two effects, shifting both the $S^*/n$ curve to the left and also shifting the constrained portion of $m$ to the left, since the representative firm's demand constraint, $C/n$, tightens. If an equilibrium exists, these two curves will eventually cross at a point on the $a$-curve (not shown). Clearly, this is in the region of increasing returns.

\[ \begin{align*}
S^*/n &< S*/n^* \\
S^*/n &< S^*/n^*
\end{align*} \]

Fig. 4. Variable returns to scale.

The appendix to this section provides a more formal demonstration that $dn/d\bar{C} < 0$ in the vicinity of the Walrasian solution, quite the opposite of the optimal response of $n$, as well as one's casual impressions. Thus, with variable returns to scale, free entry eliminates the paradox of countercyclical prospective profits, but only by introducing a new paradox, of countercyclical firm population. The appendix shows further paradoxes as well. If the elasticity of substitution between current and future leisure and consumption is less than unity, then investment and employment are countercyclical, as the income effects of pessimism lead consumers to reduce leisure (increasing employment) and current consumption (increasing savings and investment). In any case, it can also be shown (details available upon request) that investment and employment are greater than in the constrained optimum, as resources are used inefficiently in producing $\bar{C}$ at too small a scale.

IV. DISSIPATION OF PROSPECTIVE PROFITS BY DUP ACTIVITIES

Bhagwati (1982) defines directly unproductive profit-seeking (DUP) activities as those activities which 'yield pecuniary returns but do not produce goods or services that enter a utility function directly or indirectly... Insofar as such
activities use real resources, they result in a contraction of the availability set open to the economy’ (p. 989). Although most of the rent-seeking literature focuses on activities arising out of government policies (licensing, regulation, etc.), Bhagwati notes that some ‘exclusively private’ (p. 991) activities may fall in the same category. Gordon (1982) analyses data on ‘entrepreneurial activities’, which comprise over a quarter of U.S. manufacturing revenues. He uses a technique of productivity accounting to try to infer what portion of these activities (which include marketing, product development, etc.) are unproductive. He concludes that either ‘relatively little of the increased entrepreneurial activity in the manufacturing sector was devoted to increasing productivity or that massive events beyond the control of management were operating to reduce productivity’ (p. 494).

In the present paper, prospective profits arising from the classical interest response to demand constraints are a clear incentive to DUP activities. In particular, combative marketing activities directed at profitable shares of a fixed market, \( \tilde{C} \), are certainly unproductive (leaving aside the informational and distributive services provided by some portion of such activities). In Bhagwati’s taxonomy, such activities are ‘necessarily immiserising’ (p. 922). He cites the example of mutually deterring lobbying that does not affect policy, but which uses up resources. The analogy with combative marketing activity is evident.

Formally, firm \( f \)'s period \( t \) profits are given by

\[
\Pi_f^t = F(K^t, L^t) - wL^t - rK^t - D^t,
\]

where \( D \) denotes DUP activities to be undertaken in period \( t \). Suppose the market share of firm \( f \) depends on its own DUP activities and those of its rivals as follows:

\[
\text{market share} = \frac{h(D_f)}{H}, \quad \text{where} \quad H = \sum_f h(D_f),
\]

and where \( h \) is S-shaped.

Also note that in this model, aggregate demand is composed of not only consumption \( \tilde{C} \), but DUP activities \( D \) as well, since, by definition, they use up resources. That is, our period \( t \) national income equation is

\[
F(K, L) = \tilde{C} + D, \quad (16)
\]

where we have, for simplicity, restored the assumption of constant returns, so \( n \) need not enter (16).

On the firm level, form the Lagrangian,

\[
F(K^t, L^t) - wL^t - rK^t - D^t + \lambda[(\tilde{C} + D) \frac{h(D_f)}{H} - F(K^t, L^t)],
\]

where \( h(D_f)/H \) is firm \( f \)'s share of aggregate demand \( (\tilde{C} + D) \). The first-order conditions for capital and labour inputs are

\[
r = (1 - \lambda) F_K \quad \text{and} \quad w = (1 - \lambda) F_L, \quad (17)
\]

as in previous sections. For DUP activities, we have the first-order condition

\[
\lambda(\tilde{C} + D) h' (D_f)/H = 1, \quad (18)
\]

1 See Colander (1984) for recent contributions and for a bibliography.
where we have assumed that each firm behaves according to the ‘large group’ assumption that the market variables $H$ and $D$ are independent of its activities.

Prospective profits lead to entry, raising $H$, which tightens each firm’s constraint, $(\bar{C} + D) h/H$, and reduces each firm’s operating revenues per dollar of DUP activities. Ultimately prospective profits are dissipated. At this point, we have aggregate DUP activities

$$
D = F(K, L) - wL - rK
= (F_K - r) K + (F_L - w) L
= \lambda F,
$$

$$
D = \lambda (\bar{C} + D),
$$

where we have used Euler’s theorem, (17), and (16). Substituting into (18), and using the equilibrium relations $D = nD^f$, $H = nh(D^f)$, we have

$$
i = Dh’(D^f)/nh(D^f)
= D^f h’(D^f)/h(D^f),
$$

$$
i = \theta(D^f),
$$

where $\theta(D^f)$ is the elasticity of $h$ with respect to $D^f$.

Gathering our analysis of this section together, the market solution for $C_o$, $L_o$, $L_1$, $K_1$, $D$, and $n$ is given by

$$
-U_{L_o}/U_{C_o} = F_{L_o} \quad (19)
$$

$$
-U_{L_1}/U_{C_0} = F_{L_1}/F_{K_1} \quad (20)
$$

$$
F_o = C_o + K_1 - \delta K_0 \quad (21)
$$

$$
(-U_{L_1}/U_{C_1}) L_1 + (U_{C_o}/U_{C_1}) K_1 = \bar{C} \quad (22)
$$

$$
F_1 = \bar{C} + D \quad (23)
$$

$$
\theta(D/n) = 1. \quad (24)
$$

This system is decomposable: (19)–(22) solve for $C_o$, $L_0$, $L_1$, and $K_1$, conditional on $\bar{C}$. This, of course, suffices to solve for the community’s welfare, $U(C_o, L_o, \bar{C}, L_1)$. Then (23) only serves to show what level of waste, $D$, is implied by the solution, and (24) solves for $n$, which is of less direct interest in this model. (This also means that the form of $h(D^f)$ and $\theta(D^f)$ have no real effects.)

The subsystem (19)–(22) is actually identical to that of Section III. Also, in both models, prospective profits are dissipated by the inefficient proliferation of firms. The only difference is in the form of the inefficiency. In Section III, the proliferation of firms reduces the productivity of each firm. Here, the waste per firm, $D^f$, is given by (24). The proliferation of firms simply proliferates the waste, increasing aggregate DUP activities, $D$.

Again, we have eliminated the paradox of countercyclical profits, but only by introducing countercyclical firm population and marketing activity. The other paradoxes discussed at the end of Section III hold here as well.
In Sections III and IV we saw two mechanisms by which the interest rate and the average return to capital could be equilibrated. Each mechanism involved the proliferation of firms and reduced the efficiency of each firm, thus bringing the average return to capital down to the interest rate.

Alternatively, consider how a competitive capital market would equilibrate the interest rate with the average return to capital. If the interest rate were to fall below the average return to capital,

\[ r_1 < \frac{F(K'_1, L'_1) - w_1 L'_1}{K'_1}, \]

then the stock market value of the firm (discounted quasi-rents),

\[ \frac{F(K'_1, L'_1) - w_1 L'_1}{r_1}, \]

would exceed the replacement cost, \( K'_1 \) (Tobin's \( q \) would exceed unity). This would open up arbitrage possibilities. Financiers could borrow at \( r_1 \), start up firms at replacement cost \( K'_1 \), sell them off in the stock market, loan the proceeds out at \( r_1 \), and in period 1 net out \( IV'_1 = F(K'_1, L'_1) - w_1 L'_1 - r_1 K'_1 \). Clearly such arbitrage possibilities would lead to indefinitely large borrowing, preventing the interest rate from falling below the average return to capital.

The mechanism just described gives us the Keynesian result that the interest rate need not adjust to equilibrate investment to full-employment, full-capacity savings, and we get this result without resorting to liquidity traps or wage/price inflexibility. Formally, the arbitrage (zero-profit) condition provides an additional equation to the classical system (1)-(2), (6)-(11), which is inconsistent with it, according to (12). Following Keynesian analysis, high interest rates depress investment and, hence, current employment, so (6) and/or (9) fail, leaving unemployment and/or excess capacity. In this way, pessimism about the future spills over into perceived constraints in the present, by workers and/or firms. Of course, this means the optimality condition (3) also fails,

\[ F_{L_0} < -\frac{U_{L_0}}{U_{C_0}}, \]

so the efficiency loss also spills over to the current period.

Formally, the zero-profit condition for period 1, together with cost-minimisation, and Euler's theorem imply future marginal productivity payments. (In a sense, this turns Say's Law on its head: future demand creates its own supply, and no more.) If consumers perceive no future constraints (particularly on employment), then we have

\[ \frac{-U_{L_1}}{U_{C_1}} = w_1 = F_{L_1}, \]

\[ \frac{U_{C_0}}{U_{C_1}} = r_1 = F_{K_1}. \]

\(^1\) For a model of uncertainty in a similar vein, see Costrell (1983).
This system, together with the national income equations

\[ F_0 = C_0 + K_1 - \delta K_0, \]
\[ F_1 = \bar{C}, \]

solve for \( C_0, L_0, L_1, K_1, w_1, \) and \( r_1. \)

Note that the quantity of current employment is determined by this system, even though the current wage is not (since we have dropped both (6) and (9)). Fig. 5 illustrates the current labour market. In the *General Theory*, Keynes (1964) retained (9), dropping (6), reaching point A on the diagram, with involuntary unemployment. Patinkin’s (1965) quantity-constrained firms accept excess capacity (dropping (9)), so their truncated labour demand curve intersects the labour supply at point \( B, \) the current full-employment wage (given by (6)). Keynes (1939) was willing to split the difference and take the wage as a constant, at point \( E. \) We need not take a position on this question here, since our main concerns are the quantities (hence, efficiency), and also the interest rate, not the current wage rate.

The behaviour of the interest rate is illustrated in the loanable funds diagram, Fig. 2. In a demand-constrained regime, the classical, full-employment interest rate is depicted at \( C'. \) The zero-profit equilibrium, however, requires a higher interest rate, which depresses investment. In Keynesian fashion, current employment is depressed as well, shifting the savings curve left until it intersects the marginal efficiency curve at \( B',1 \) eliminating the inframarginal trapezoid.

To summarise, then, the Keynesian mechanism does avoid the paradox of countercyclical profits, raised by the basic classical model of Section II. Furthermore, the appendix shows that in the Keynesian model, investment and current employment are unambiguously procyclical, as we would expect, unlike the zero-profit classical models of Sections III and IV. Finally, it can be shown that the Keynesian model gives us lower investment and current employment than in the constrained optimum, unlike the previous models.

1 This somewhat oversimplifies the comparison between the two models, since \( w_1 \) will differ between them as well, and, hence, the heights of the marginal efficiency curves. Still, it can be shown formally that the Keynesian interest rate does exceed the classical one.
CONCLUSION
We have considered the classical position that Keynesian pessimism about the future need not compromise the present, because interest rates can adjust to clear current markets. If so, we have shown that this would indeed be a constrained optimum. However, we have also shown that this second-best configuration requires interest to fall below the marginal product of capital, yielding prospective profits. This is not a sustainable competitive equilibrium. In avoiding the inefficiency of Keynesian underemployment, due to inadequate investment (Section V), low interest rates can lead to other, profit-induced, inefficiencies (Sections III and IV). Therefore, we reluctantly conclude that low interest rates are not an unmitigated blessing, unless they can be somehow coupled with effective management of expected future demand.

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To simplify the comparative statics, and to reduce the ambiguity of their signs, we assume in this appendix that the utility function is CES:

\[ U = [k_0 C_0^\rho + l_0 (\bar{L} - L_0)^{-\rho} + k_1 C_1^\rho + l_1 (\bar{L} - L_1)^{-\rho}]^{-1/\rho}, \]

where \( \bar{L} \) is the maximum leisure available. Then, for elasticity of substitution \( \sigma \equiv 1/(1 + \rho) \),

\[ \frac{U_{C_i}}{U_{C_j}} = k_{ij} (C_j/C_i)^{1/\sigma}, \quad -\frac{U_{L_i}}{U_{C_j}} = l_{ij} [C_j/(\bar{L} - L_i)]^{1/\sigma}, \]

where \( k_{ij} \) and \( l_{ij} \) are constants. The partial derivatives of these marginal rates of substitution are:

\[ \frac{\partial (U_{C_i}/U_{C_j})}{\partial C_i} = -\frac{(U_{C_i}/U_{C_j})}{\sigma C_i} < 0 \]
\[ \frac{\partial (U_{C_i}/U_{C_j})}{\partial C_j} = \frac{(U_{C_i}/U_{C_j})}{\sigma C_j} > 0 \]
\[ \frac{\partial (-U_{L_i}/U_{C_j})}{\partial L_i} = \frac{-U_{L_i}/U_{C_j}}{\sigma (\bar{L} - L_i)} > 0 \]
\[ \frac{\partial (-U_{L_i}/U_{C_j})}{\partial C_j} = \frac{-U_{L_i}/U_{C_j}}{-C_j} > 0. \]

Also denote the marginal rate of technical substitution between \( K_1 \) and \( L_1 \), 

\[ F_{L_1}/F_{K_1} = T(K_1, L_1), \]

where \( T_K > 0 \) and \( T_L < 0 \).

We confine our attention to Range (ii) of \( \bar{C} \), where \( \mu_2 > 0 \) and \( \mu_3 = 0 \). Totally differentiating the system (3), (5), (1), and \( F(K_1, L_1) = \bar{C} \), we have

\[ J [dC_0 \ dL_0 \ dL_1 \ dK_1] = [0 \ 0 \ 0 \ 0]' d\bar{C}, \]

where

\[ J = \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ J_{21} & 0 & J_{23} & -T_K \\ 1 & -F_{L_0} & 0 & 1 \\ 0 & 0 & F_L & F_K \end{bmatrix}. \]

Here,

\[ J_{11} = F_{L_0}/\sigma C_0 > 0, \]
\[ J_{12} = F_{L_0}/\sigma (\bar{L} - L_0) - F_{L_0} > 0, \]
\[ J_{21} = T(K, L)/\sigma C_0 > 0, \]
\[ J_{23} = T(K, L)/\sigma (\bar{L} - L_1) - T_L > 0, \]

and where \( F \) is evaluated at period 1 values unless otherwise indicated, to economise on notation.

First, evaluate the minors excluding the fourth row:

\[ |J_{41}| = J_{12} J_{23} > 0, \]
\[ |J_{42}| = J_{11} J_{23} > 0, \]
\[ |J_{43}| = -J_{12} T_K - J_{12} J_{21} - J_{11} T_K F_{L_0} < 0, \]
\[ |J_{44}| = J_{12} J_{23} + J_{11} J_{23} F_{L_0} > 0. \]
Then \( \det J = F_K|J_{44}| - F_L|J_{43}| > 0 \). The comparative static derivatives for the quantities are immediate, using Cramer's rule:

\[
\begin{align*}
dC_0/d\bar{C} &= -\frac{|J_{41}|}{|J|} < 0, \\
dL_0/d\bar{C} &= \frac{|J_{42}|}{|J|} > 0, \\
dL_1/d\bar{C} &= -\frac{|J_{43}|}{|J|} > 0, \\
dK_1/d\bar{C} &= \frac{|J_{44}|}{|J|} > 0.
\end{align*}
\]

Thus, as \( \bar{C} \) falls, leisure and current consumption should be substituted for future consumption.

Section II's required factor price response for \( w_0 \) is found by differentiating (9):

\[
dw_0/d\bar{C} = F_{L0} dL_0/d\bar{C} < 0.
\]

The required future factor price responses are found by differentiating (7) and (8):

\[
\begin{align*}
dw_1/d\bar{C} &= \left[ w_1/\sigma(L - L_1) \right] dL_1/d\bar{C} + w_1/\sigma\bar{C} > 0, \\
dr_1/d\bar{C} &= - (r_1/\sigma C_0) dC_0/d\bar{C} + r_1/\sigma\bar{C} > 0.
\end{align*}
\]

As \( \bar{C} \) falls, the future wage and interest rate must both fall to induce consumers to enjoy more future leisure and current consumption.

### Appendix to Section III

Consider first the optimal solution for \( C_0, L_0, L_1, K_1, \) and \( n_1 \). It is given by the system

\[
\begin{align*}
-U_{L0}/U_{C0} &= F_{L0}, \quad (i) \\
-U_{L1}/U_{C0} &= T(K_1/n_1, L_1/n_1) = T(K_1L_1), \quad \text{by the assumption of homotheticity}, \quad (ii) \\
n_0 F(K_0/n_0, L_0/n_0) &= C_0 + K_1 - \delta K_0, \quad (iii) \\
n_1 F(K_1/n_1, L_1/n_1) &= \bar{C}, \quad (iv) \\
n_1 F(K_1/n_1, L_1/n_1) &= K_1 F_{K_1} + L_1 F_{L_1}. \quad (v)
\end{align*}
\]

Let the default subscript indicate period 1. Totally differentiate (iv) to find

\[
F_K dK + F_L dL + (F - KF_K/n - LF_L/n) dn = d\bar{C}. \quad (vi)
\]

By (v), this reduces to

\[
F_K dK + F_L dL = d\bar{C}, \quad (vii)
\]

the same as in the linear homogeneous case of Section I. Thus, the differential system of (i)–(iv) forms a subsystem in \( dC_0, dL_0, dL_1, \) and \( dK_1 \), given \( d\bar{C} \), which is identical to that analysed in the previous appendix. Therefore, the comparative static derivatives of these variables are the same.

The only other comparative static derivative to evaluate for the optimum is \( dn/d\bar{C} \). The following proof shows the intuitive result \( dn/n = d\bar{C}/\bar{C} \), i.e. the
optimum output per firm stays constant (though the factor proportions do not, in general) as demand varies. To see this, equate (iv) to (v) and differentiate:

\[(F_K + KF_{KK}/n + LF_{KL}/n) dK + (F_L + LF_{LL}/n + KF_{KL}/n) dL - (K^2F_{KK} + KLF_{KL} + KLF_{KL} + L^2F_{LL}) dn/n^2 = d\bar{C}. \quad (viii)\]

For our homothetic production function, we have:

\[
F_{KK} = g''(G_K)^2 + g'G_{KK},
\]
\[
F_{LL} = g''(G_L)^2 + g'G_{LL},
\]
\[
F_{KL} = g''G_KG_L + g'G_{KL}.
\]

Therefore,

\[
KF_{KK} + LF_{KL} = g''G_K(KG_K + LG_L) + g'(KG_{KK} + LG_{KL}) = g''G_KG,
\]

using the linear homogeneity of \(G\). Similarly,

\[
LF_{LL} + KF_{KL} = g''G_LG.
\]

Substituting into (viii), we have

\[
F_K dK + F_L dL + (g''G/n) (G_K dK + G_L dL) = d\bar{C} + (g''G) (KG_K + LG_L) dn/n^2.
\]

Using (vii) and rearranging yields

\[
\frac{dn}{n} = \frac{G_K dK + G_L dL}{KG_K + LG_L}.
\]

Multiplying through the right-hand side by \(g'/g'\) gives us,

\[
\frac{dn}{n} = \frac{F_K dK + F_L dL}{KF_K + LF_L} = \frac{d\bar{C}}{\bar{C}},
\]

using (iv), (v), and (viii), as stated.

Now let us turn to the zero-profit market solution. Using (7) and (8), the zero-profit condition is

\[
o = \Pi = nF - wL - rK
\]
\[
= \bar{C} + (U_{L_1}/U_{C_1}) L - (U_{C_0}/U_{C_1}) K,
\]
\[
or \quad (- U_{L_1}/U_{C_1}) L + (U_{C_0}/U_{C_1}) K = \bar{C}. \quad (ix)
\]

The expressions (i)–(iii) still hold, and, together with (ix), form a subsystem which solves for \(C_0, L_0, L_1,\) and \(K_1\), conditional on \(\bar{C}\). Since (i)–(iii) are also (3), (5), and (i), this subsystem differs from the optimal system only by (ix), the zero-profit condition. Totally differentiating the subsystem, we have

\[
J [dC_0, dL_0, dL_1, dK_1]' = B [d\bar{C}],
\]

where the first three rows of \(J\) were calculated in the previous appendix. We have

\[
J = \begin{bmatrix}
J_{11} & J_{12} & 0 & 0 \\
J_{21} & 0 & J_{23} & -T_K \\
J_{41} & 0 & J_{43} & (1-t)r
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
0 \\
B_{41}
\end{bmatrix},
\]
where $J_{11}, J_{12}, J_{21},$ and $J_{43}$ are positive numbers given in the previous appendix, $J_{41} = -Kr/\sigma C_0 < 0,$ and $J_{43} = w_1[1 + L_1/\sigma(\bar{L} - L_1)] > 0.$ Also,

$$B_{41} = (1 - w_1 L_1/\sigma \bar{C} - rK/\sigma \bar{C})$$

$$= (\sigma \bar{C} - \bar{C})/\sigma \bar{C},$$  using the zero-profit condition,

$$= (\sigma - 1)/\sigma \geq 0 \text{ as } \sigma \geq 1.$$

Note that the minors excluding the fourth row are the same as in the previous appendix. Therefore,

$$|J| = -J_{41} |J_{41}| - J_{43} |J_{43}| + r |J_{44}| > 0.$$

The comparative static derivatives then follow immediately:

$$dC_0/d\bar{C} = -B_{41} |J_{41}|/|J| \leq 0 \text{ as } \sigma \geq 1,$$

$$dL_0/d\bar{C} = B_{41} |J_{42}|/|J| \geq 0 \text{ as } \sigma \geq 1,$$

$$dL_1/d\bar{C} = -B_{41} |J_{43}|/|J| \geq 0 \text{ as } \sigma \geq 1,$$

$$dK_1/d\bar{C} = B_{41} |J_{44}|/|J| \geq 0 \text{ as } \sigma \geq 1.$$

Thus, if we compare these with the optimal responses, given in the previous appendix, we find that the variables here move in the same direction (though presumably not as far) if and only if $\sigma > 1.$ The same holds for $w_0,$ which depends only on $L_0.$ For $w_1$ and $r,$ differentiate (7) and (8) to find

$$dw_1/d\bar{C} = [w_1/\sigma(\bar{L} - L_1)] dL_1/d\bar{C} + w_1/\sigma \bar{C}$$

$$dr/d\bar{C} = -(r/\sigma C_0) dC_0/d\bar{C} + r/\sigma \bar{C},$$

so if $\sigma > 1,$ $dw_1/d\bar{C}$ and $dr/d\bar{C} > 0,$ as in the previous appendix. But this also holds for $\sigma = 1,$ and, presumably, for at least some range below unity. We have not been able to provide a global proof, however.

Now we will show that $dN/d\bar{C} < 0$ in the vicinity of the unconstrained solution.

From (vi) we have

$$dn/d\bar{C} = (F_K dK/d\bar{C} + F_L dL/d\bar{C} - 1)/(e - 1) F.$$  \hfill (x)

Since we are in the constrained region, $\lambda > 0,$ and, as the text shows, $e > 1.$ Therefore, the denominator is positive (although it vanishes as $\bar{C}$ approaches the unconstrained value of $C_1,$ so $dn/d\bar{C}$ becomes infinite at this point). For $\sigma < 1,$ $dK/d\bar{C}$ and $dL/d\bar{C}$ are negative, hence so is $dn/d\bar{C}.$ For $\sigma > 1,$ we have $0 < B_{41} < 1,$ so

$$dK/d\bar{C} = B_{41} |J_{44}|/|J| < |J_{44}|/|J|,$$

$$dL/d\bar{C} = -B_{41} |J_{43}|/|J| < -|J_{43}|/|J|$$

and

$$F_K dK/d\bar{C} + F_L dL/d\bar{C} < (F_K |J_{44}| - F_L |J_{43}|)/|J|.  \hfill (xi)$$

Recall that

$$|J| = |r |J_{44}| - w_1 |J_{43}| - w_1 L_1 |J_{45}|/\sigma(\bar{L} - L_1) + rK|J_{41}|/\sigma C_0|.$$  

In the vicinity of the unconstrained solution, $r \rightarrow F_K$ and $w_1 \rightarrow F_L.$ Thus, in this vicinity, the difference between the numerator and denominator of the right-hand
side in (xi) is dominated by the last two terms of |\(J_I|\), both of which are positive. Therefore, in this vicinity, at least, \(F_K dK/d\bar{C} + F_L dL/d\bar{C} < 1\), so, by (x), \(dn/d\bar{C} < 0\) regardless of \(\sigma\). This, of course, is the opposite of the optimal response, \(dn/d\bar{C} = n/\bar{C} > 0\).

**APPENDIX TO SECTION V**

The quantity system is decomposable:

\[
-UL_1/U_{C_1} = F_{L_1},
\]

(i)

\[
UC_0/U_{C_1} = F_{K_1},
\]

(ii)

\[
F_1 = \bar{C},
\]

(iii)
solve for \(C_0, L_1,\) and \(K_1,\) conditional on \(\bar{C}\). The differential system is

\[
J \begin{bmatrix} dC_0 \\ dL_1 \\ dK_1 \end{bmatrix}' = B \begin{bmatrix} d\bar{C} \end{bmatrix},
\]

where

\[
J = \begin{bmatrix}
0 & J_{12} & -F_{K_1} \\
-F_K/\sigma C_0 & -F_{K_L} & -F_{K_K} \\
0 & F_L & F_K
\end{bmatrix}, \quad B = \begin{bmatrix}
-F_L/\sigma \bar{C} \\
-F_K/\sigma \bar{C} \\
1
\end{bmatrix}.
\]

Here,

\[
J_{12} = F_L/\sigma (L - L) - F_{LL} > 0.
\]

Then

\[
|J| = [F_{KL} F_K F_L/\sigma C_0 + F_{KL}^2 J_{12}/\sigma C_0] > 0.
\]

The comparative statics with respect to \(\bar{C}\) tell us that current consumption is procyclical:

\[
|J| dC_0/d\bar{C} = 2F_L F_K F_{KL}/\sigma \bar{C} - F_{KK} F_L/\sigma (L - L)
+ F_{KK} F_{LL} - F_{L}^2 F_{KL} - F_{L}^2 F_{KK}/\sigma \bar{C} + F_{L}^2 J_{12}/\sigma \bar{C} > 0,
\]

since \(F_{LL} F_{KK} = F_{KL}^2\) under constant returns.

Future employment is ambiguous:

\[
|J| dL_1/d\bar{C} = F_K F_{KL}/\sigma C_0 - F_{K}^2 F_L/\sigma^2 C_0 \bar{C}
= (F_K/\sigma^2 C_0) (\sigma F_{KL} - F_K F_L/F)
= (F_K/\sigma^2 C_0) F_{KL} (\sigma - \sigma_p),
\]

where \(\sigma_p = F_K F_L/FF_{KL}\) is the elasticity of substitution in the production function. Thus, \(dL_1/d\bar{C} \geq 0\) as \(\sigma \geq \sigma_p\).

Investment, however, is unambiguously procyclical:

\[
|J| dK/d\bar{C} = F_L^2 F_K/\sigma^2 C_0 \bar{C} + J_{12} F_K/\sigma C_0 > 0.
\]

Also, from (27), current employment is procyclical:

\[
dL_0/d\bar{C} = (dC_0/d\bar{C} + dK/d\bar{C})/F_{L_0} > 0.
\]
The factor costs (and rewards) vary with the marginal productivities, and the marginal productivities depend only on the capital–labour ratio. We have

\[
\text{sgn} \frac{d(K/L)}{d\bar{C}} = \text{sgn} \frac{LdK/d\bar{C} - KdL/d\bar{C}}{dC} = \text{sgn} |J| \left( \frac{LdK/d\bar{C} - KdL/d\bar{C}}{dC} \right) = \frac{LF_L^2 F_K/\sigma^2 C_0 \bar{C} + LF_K F_L/\sigma^2 (\bar{L} - L) C_0}{C + (LF_K F_{LL}/\sigma C_0 + KF_K F_{KL}/\sigma C_0) + KF_K F_{L}/\sigma^2 C_0 \bar{C}} > 0,
\]

since the term in parentheses vanishes under linear homogeneity. Thus, the marginal productivity of labour rises with \( \bar{C} \), as does \( w_1 \), while \( F_K \) and \( r_1 \) fall.